ECE 359 **Spring 2003**

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NOTES ON RANDOM PROCESSES – II

Stationarity: A random process is said to be strictly stationary if, for each n, and each choice of $t_1, t_2, \ldots, t_n$, the joint cdf of $X(t_1), X(t_2), \ldots, X(t_n)$ is the same as the joint cdf of $X(t_1+t), X(t_2+t), \ldots, X(t_n+t)$, for any t. That is, the statistics of the random process are invariant to time shifts.

Strict sense stationarity is a very strong condition to require on a random process. In practice, it is enough if only first and second order conditions are satisfied.

Wide Sense Stationarity: A random process $X(t)$ is said to be wide sense stationary (WSS) if its mean does not change with time and its autocorrelation function depends on only the time difference between the samples. That is

$$m_X(t) = m_X \qquad \text{and} \qquad R_X(t+\tau, t) = R_X(\tau)$$

We will mainly be concerned with WSS processes in this course. Note that for a WSS process

$$\mathrm{E}[X(t_1)X(t_2)] = \mathrm{E}[X(t_2 + (t_1 - t_2))X(t_2)] = R_X(t_1 - t_2)$$

If a process is strictly stationary, it is also WSS – of course, the converse is not true.

Example 2 (continued): For the sinusoid with random phase, we see that $m_X(t)$ and $R_X(t+\tau, t)$ are indeed independent of t. Hence this random process is WSS, with $m_X = 0$ and $R_X(\tau) = a^2\cos(2\pi f_0 \tau)/2$

Some Properties of the Autocorrelation Function of a WSS Process:

1) $R_X(\tau)$ is an even function of τ.
2) $R_X(0) \geq 0$. (Since the secont moment of $X(t)$ is ≥ 0.)
3) R_X acheives its maximum absolute value at 0, i.e., $|R_X(\tau)| \leq R_X(0)$ (by the Cauchy-Schwarz inequality).

Example 3 A random process $X(t)$ is defined by

$$X(t) = A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)$$

where A and B are random variables such that $m_A = m_B = 0$, $\mathrm{E}[AB] = 0$, and $\sigma_A^2 = \sigma_B^2 = \sigma^2$. We showed in class that this random process is WSS with $m_X = 0$ and $R_X(\tau) = \sigma^2\cos(2\pi f_0 \tau)$.

Two Random Processes:

Let $X(t)$ and $Y(t)$ be two random processes. Just as in the case of two random variables, we are interested in the *joint* statistical properties of these processes.

Uncorrelated Processes

The two processes are said to be *uncorrelated* if for any t_1 and t_2

$$\mathrm{E}[X(t_1)Y(t_2)] = \mathrm{E}[X(t_1)]\,\mathrm{E}[Y(t_2)].$$

Crosscorrelation Function

The crosscorrelation function for the two processes $X(t)$ and $Y(t)$ is defined by

$$R_{X,Y}(t+\tau,t) = \mathrm{E}[X(t+\tau)Y(t)].$$

Note that $R_{X,Y}(t+\tau,t) = R_{Y,X}(t,t+\tau)$, but $R_{X,Y}(t+\tau,t) \neq R_{Y,X}(t+\tau,t)$

Jointly WSS Processes

Two processes $X(t)$ and $Y(t)$ are said to be jointly WSS if: (a) $X(t)$ is WSS; (b) $Y(t)$ is WSS; and (c) $R_{X,Y}(t+\tau,t) = R_{X,Y}(\tau)$. Note that **all three** conditions must be satisfied.

Properties of the Cross-correlation Function of Jointly WSS Processes

(a) $R_{X,Y}(-\tau) = R_{Y,X}(\tau)$. (Unlike the ACF, the crosscorrelation function is not even.)

(b) $|R_{X,Y}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$. (Follows from the Cauchy-Schwarz inequality.)

(c) If two jointly WSS processes are uncorrelated, then $R_{X,Y}(\tau) = \mathrm{E}[X(t)]\mathrm{E}[Y(t+\tau)] = m_X m_Y$

Example 4 Consider two random processes $X(t)$ and $Y(t)$ defined by

$$\begin{aligned} X(t) &= A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t) \\ Y(t) &= B\cos(2\pi f_0 t) - A\sin(2\pi f_0 t) \end{aligned}$$

where A and B are random variables such that $m_A = m_B = 0$, $\mathrm{E}[AB] = 0$, and $\sigma_A^2 = \sigma_B^2 = \sigma^2$.

We showed earlier (see Example 3) that $X(t)$ is WSS with $m_X = 0$ and $R_X(\tau) = \sigma^2\cos(2\pi f_0\tau)$. It is easy to show that $Y(t)$ is also WSS with $m_Y = 0$ and $R_Y(\tau) = \sigma^2\cos(2\pi f_0\tau)$. Now let us consider the cross-correlation. As we showed in class

$$R_{X,Y}(t+\tau,t) = \sigma^2\sin(2\pi f_0\tau)$$

Thus $R_{X,Y}(t+\tau,t)$ is independent of t. This, together with the fact that both $X(t)$ and $Y(t)$ are WSS, implies that the two processes are jointly WSS.

Spectral Characterization of Random Processes

For a random process $X(t)$, the Fourier transform of the autocorrelation function $R_X(\tau)$, denoted by $S_X(f)$, is called the *power spectral density* of the process $X(t)$. Thus

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}d\tau.$$

The power spectral density and autocorrelation form a Fourier transform pair, $R_X(\tau) \rightleftharpoons S_X(f)$. So we get

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{j2\pi f\tau}df.$$

In particular,

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)df. \tag{1}$$

Note that for any random process $X(t)$, $\mathrm{E}[\{X(t)\}^2]$ is the *ensemble* average of $X^2(t)$ at time t. If the process is WSS, then $\mathrm{E}[\{X(t)\}^2] = R_X(0)$, for all t. Furthermore, if we assume that the process is *ergodic*, then $\mathrm{E}[\{X(t)\}^2]$ also equals the time average of $X^2(t)$ over all sample paths of $X(t)$. Thus $R_X(0)$ represents the *average power* in a WSS process, and the relationship given in (1) further clarifies why $S_X(f)$ is called a power spectral density.

Properties of the Power Spectral Density

(a) The power in $X(t)$ between the frequencies $[-f_0, f_0+\Delta f]$ and $[-f_0, -f_0-\Delta f]$ is approximately equal to $2S(f_0)\Delta f$ (for sufficiently small Δf)

(b) $S_X(f)$ is a real-valued even function, and $S_X(f) \geq 0$ for all f.

LINEAR FILTERING OF RANDOM PROCESSES

Recall that if a deterministic signal $x(t)$ is passed through a LTI system with impulse response $h(t)$, the output $y(t)$ is given by $y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = x(t) \star h(t)$. The question that we now ask is: What happens when a random process $X(t)$ is passed through a LTI system?

First of all, since the input to the system is a random process, the output (except in some trivial cases) is also going to be a random process which we will denote by $Y(t)$. Now, using the definition of impulse response and following a reasoning similar to the deterministic signal case, we have that

$$Y(t) = \int_{-\infty}^{\infty} h(\lambda)X(t-\lambda)d\lambda.$$

Now suppose that $X(t)$ is a WSS process with mean m_X and autocorrelation function $R_X(\tau)$. We are interested in finding the mean and autocorrelation function of $Y(t)$. We cannot assume that $Y(t)$ is WSS – we have to prove that it is. We can show the following (we did this in class):

$$m_Y(t) = m_X \int_{-\infty}^{\infty} h(\lambda)d\lambda = m_X H(0) = m_Y \quad \longleftarrow \quad \text{independent of } t$$

where $H(f)$ is the Fourier transform of $h(t)$,

$$R_{X,Y}(t+\tau, t) = \int_{-\infty}^{\infty} h(\lambda)R_X(\tau+\lambda)d\lambda = R_X(\tau) \star h(-\tau) = R_{X,Y}(\tau) \quad \longleftarrow \quad \text{independent of } t$$

and

$$\begin{aligned} R_Y(t+\tau, t) &= \int_{-\infty}^{\infty} h(\lambda)R_{XY}(\tau-\lambda)d\lambda = R_{X,Y}(\tau) \star h(\tau) \\ &= R_X(\tau) \star h(-\tau) \star h(\tau) = R_Y(\tau) \quad \longleftarrow \quad \text{independent of } t \end{aligned}$$

Note that the above equations imply the following for a LTI system:

If the input $X(t)$ is WSS, the output $Y(t)$ is also WSS; also, $X(t)$ and $Y(t)$ are jointly WSS

The power spectral density of $Y(t)$ can be expressed compactly in terms of the power spectral density of $X(t)$. We have

$$S_Y(f) = \mathcal{F}\{R_X(\tau) \star h(\tau) \star h(-\tau)\} = S_X(f)\, H(f)\, \mathcal{F}\{h(-\tau)\}$$

where $\mathcal{F}$ denotes the Fourier transform operation. It is easy to show that $\mathcal{F}\{h(-\tau)\} = H^*(f)$. Thus

$$S_Y(f) = H(f)\, H^*(f)\, S_X(f) = |H(f)|^2\, S_X(f)$$