ECE 359 **Spring 2003**

Handout # 13 March 20, 2003

GAUSSIAN RANDOM PROCESSES

A process $X(t)$ is said to be a Gaussian process if for every n, and every choice of $t_1, t_2, \ldots, t_n$, the joint distribution of the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ is *jointly* Gaussian. (See Handout #9 for a discussion of jointly Gaussian random variables.) Many processes that arise from natural phenomena can be approximated well by Gaussian processes, using Central Limit Theorem arguments. Gaussian processes have the following nice properties:

- A Gaussian process is *completely* characterized by its mean and autocorrelation function.
- A Gaussian process that is WSS is also *strictly* stationary
- If a Gaussian process is passed through a linear system, the output is also a Gaussian process.

We will now consider the Gaussian process that is most important for communication system analysis.

White Gaussian Noise Every electrical/electronic system has inherent noise sources. This noise is caused by the chaotic motion of electrons in the components of the system, and is sometimes referred to as *thermal noise*. Experiments conducted by Johnson (and verified analytically by Nyquist) in the 1920's showed that the power spectral density of thermal noise was flat over a large range of frequencies (as high as 10^{13} Hz).

If the PSD was indeed constant over all frequencies (however large), then the random process $N(t)$ representing the thermal noise will have to have infinite power since its power spectral density $S_N(f)$ would simply be

$$S_N(f) = S_N(0) \quad \text{for all } f$$

and

$$\text{Average Power in } N(t) = R_N(0) = \int_{-\infty}^{\infty} S_N(f)df = S_N(0)\int_{-\infty}^{\infty} df = \infty.$$

However as we mentioned above, the model for thermal noise was observed to valid only for frequencies less than 10^{13} Hz. Now, no physical system has infinite bandwidth, and we can observe the thermal noise process only at the output of some physical system.

Thus we can idealize the thermal noise by a zero-mean WSS random process which has a constant power spectral density over all frequencies. This idealization is called ***white noise****.*

For an ideal white noise process $N(t)$, $m_N = 0$ and $S_N(f)$ is a constant. We will typically denote this constant by the symbol $\frac{N_0}{2}$, that is

$$S_N(f) = \frac{N_0}{2}.$$

Note that this implies that the autocorrelation function of white noise is given by

$$R_N(\tau) = \frac{N_0}{2}\delta(\tau)$$

The source of thermal noise (or, its idealization, white noise) is the chaotic motion of electrons. Since a large number of independent identical events contribute to this noise, we can apply the Central Limit Theorem to conclude that this noise is a Gaussian random process. This Gaussian random process is referred to as ideal *White Gaussian Noise* (WGN).

Filtering WGN

If we pass WGN $N(t)$ through a LTI system with impulse response $h(t)$, the output $Y(t)$ is a zero mean, (*colored*), stationary Gaussian process with

$$S_Y(f) = \frac{N_0}{2}|H(f)|^2 \quad \text{and} \quad R_Y(\tau) = \frac{N_0}{2} h(\tau) \star h(-\tau)$$

If we pass $N(t)$ through an ideal LPF with bandwidth W Hz, the output $N_{\text{LP}}(t)$ is a zero mean, stationary Gaussian process with

$$S_{N_{\text{LP}}}(f) = \frac{N_0}{2}\Pi\left(\frac{f}{2W}\right) \quad \text{and} \quad R_{N_{\text{LP}}}(\tau) = N_0 W \operatorname{sinc}(2W\tau)$$

The average power in $N_{\text{LP}}(t)$ is $N_0 W$.

If we pass $N(t)$ through an ideal BPF with center frequency f_c and bandwidth $2W$ Hz, the output $N_{\text{BP}}(t)$ is a zero mean, stationary Gaussian process with

$$S_{N_{\text{BP}}}(f) = \frac{N_0}{2}\left[\Pi\left(\frac{f-f_c}{2W}\right) + \Pi\left(\frac{f+f_c}{2W}\right)\right] \quad \text{and} \quad R_{N_{\text{BP}}}(\tau) = 2N_0 W \operatorname{sinc}(2W\tau)\cos(2\pi f_c \tau)$$

The average power in $N_{\text{BP}}(t)$ is $2N_0 W$.

Proposition The process $N_{\text{BP}}(t)$ can be expressed in terms of two low-pass processes as:

$$N_{\text{BP}}(t) = N_{\text{c}}(t)\cos 2\pi f_c t - N_{\text{s}}(t)\sin 2\pi f_c t$$

where $N_{\text{c}}(t)$ and $N_{\text{s}}(t)$ are independent, zero mean, Gaussian processes with

$$S_{N_{\text{c}}}(f) = S_{N_{\text{s}}}(f) = N_0\Pi\left(\frac{f}{2W}\right) \quad \text{and} \quad R_{N_{\text{c}}}(\tau) = R_{N_{\text{s}}}(\tau) = 2N_0 W \operatorname{sinc}(2W\tau)$$

Proof As done in class, the proof involves showing that the process $Y(t) = N_{\text{c}}(t)\cos 2\pi f_c t - N_{\text{s}}(t)\sin 2\pi f_c t$ is Gaussian, zero mean, and has the same ACF as $N_{\text{BP}}(t)$