ECE 359 **Spring 2003**

Handout # 16 April 15, 2003

SNR for ANGLE MODULATION SYSTEMS

Recall that for angle modulation:

$$v(t) = A_c \cos(2\pi f_c t + \phi(t))$$

where

$$\phi(t) = \begin{cases} 2\pi k_f \int_{-\infty}^{t} m(t')dt' & \text{for FM} \\ \\ k_p m(t) & \text{for PM} \end{cases}$$

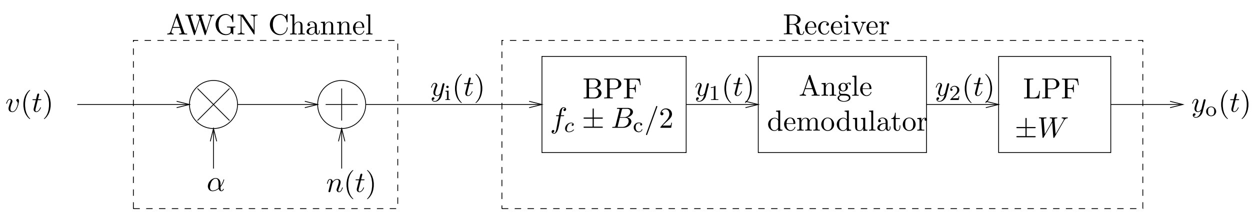
The modulation index:

$$\beta = \begin{cases} \frac{k_f m_{\max}}{W} & \text{for FM} \\ \\ k_p m_{\max} & \text{for PM} \end{cases}$$

If $\beta \ll 1$, then we have narrowband angle modulation with bandwidth $2W$ Hz. If $\beta > 5$ we have a wideband angle modulation with bandwith given by Carson's rule:

$$B_{\rm c} = 2W(1 + \beta) \text{ Hz.}$$

Now consider sending a wideband angle modulated signal on an AWGN channel as shown below.



As before, we begin the SNR analysis by writing equations for the outputs at various stages of the receiver.

$$y_i(t) = \alpha v(t) + n(t)\ , \quad \text{and} \quad y_1(t) = \alpha v(t) + n_{\rm BP}(t)$$

where $n_{\rm BP}(t)$ is a sample path of a bandpass WGN process with spectral height $N_0/2$ and bandwidth of $B_{\rm c}$ Hz centered around f_c. We can write

$$N_{\rm BP}(t) = N_c(t)\cos(2\pi f_c t) - N_s(t)\sin(2\pi f_c t)$$

where $N_c(t)$ and $N_s(t)$ are independent lowpass WGN processes with spectral height N_0 and bandwidth $B_{\rm c}/2$. We can rewrite $n_{\rm BP}(t)$ as

$$n_{\rm BP}(t) = e_n(t)\cos(2\pi f_c t + \theta_n(t))$$

where

$$e_n(t) = \sqrt{n_c^2(t) + n_s^2(t)}\ , \quad \text{and} \quad \theta_n(t) = \tan^{-1}\left(\frac{n_s(t)}{n_c(t)}\right)$$

Thus

$$y_1(t) = \alpha A_c \cos(2\pi f_c t + \phi(t)) + e_n(t)\cos(2\pi f_c t + \theta_n(t))$$

$$= r(t)\cos(2\pi f_c t + \phi(t) + n_\phi(t))$$

For small noise, i.e. $\frac{P_{\rm s,i}}{N_0 B_{\rm c}} = \frac{A_c^2\alpha^2}{2N_0B_{\rm c}} > 10$, we showed in class using phasors that

$$n_\phi(t) \approx \frac{e_n(t)}{\alpha A_c}\sin(\theta_n(t) - \phi(t))$$

Now we make the assumption that $\phi(t)$ varies slowly w.r.t. $\theta_n(t)$. This assumption holds for wideband signals since $n_c(t)$ and $n_s(t)$ have bandwidth $B_{\rm c}/2$ that is much larger than the bandwidth of $\phi(t)$ (which equals W). Thus

$$n_\phi(t) \approx \frac{e_n(t)\sin\theta_n(t)}{\alpha A_c}\cos\phi - \frac{e_n(t)\cos\theta_n(t)}{\alpha A_c}\sin\phi = \frac{n_s(t)}{\alpha A}\cos\phi - \frac{n_c(t)}{\alpha A}\sin\phi$$

This means that $N_\phi(t)$ is approximately WSS with zero mean and ACF given by

$$R_{N_\phi}(\tau) = \frac{R_{N_s}(\tau)}{\alpha^2 A_c^2}\cos^2\phi + \frac{R_{N_c}(\tau)}{\alpha^2 A_c^2}\sin^2\phi = \frac{R_{N_c}(\tau)}{\alpha^2 A_c^2}$$

i.e., $N_\phi(t)$ is a lowpass WGN process with bandwidth $B_{\rm c}/2$ Hz and spectral height $\frac{N_0}{A_c^2\alpha^2}$.

The output of the angle demodulator $y_2(t)$ is given by:

$$y_2(t) = \begin{cases} \phi(t) + n_\phi(t) = k_p m(t) + n_\phi(t) & \text{for PM} \\ \frac{1}{2\pi}\left[\phi'(t) + n'_\phi(t)\right] = k_f m(t) + \frac{1}{2\pi}n'_\phi(t) & \text{for FM} \end{cases} \qquad (1)$$

where $n'_\phi(t)$ is derivative of $n_\phi(t)$.

Output SNR for PM: For PM,

$$y_2(t) = k_p m(t) + n_\phi(t)$$

where $N_\phi(t)$ is (approximately) a WSS, zero mean, process with PSD

$$S_{N_\phi}(\omega) = \begin{cases} \frac{N_0}{A_c^2\alpha^2} & \text{if } |f| \le B_{\rm c}/2 \\ 0 & \text{otherwise} \end{cases} .$$

Recall that two key assumptions were used in arriving at this approximation for $N_\phi(t)$. These are:
① The angle modulated signal is *wideband*, i.e., $\beta > 5$
② The signal-to-noise ratio at the output of the front-end BPF is large, i.e.,

$$\frac{P_{\rm s,i}}{N_0 B_{\rm c}} = \frac{P_{\rm s,i}}{2N_0 W(\beta+1)} = \frac{\overline{\Gamma}}{2(\beta+1)} > 10\,.$$

Now

$$y_{\rm o}(t) = [y_2(t)]_{\rm LPF} = k_p m(t) + n_{\rm o}(t)$$

where $n_\text{o}(t)$ is $n_\phi(t)$ passed through a W Hz LPF filter. Since $B_\text{c} > 2W$,

$$S_{N_\text{o}}(f) = |H_\text{LPF}(f)|^2 \; S_{N_\phi}(f) = \begin{cases} \frac{N_0}{A_c^2\alpha^2} & \text{if } |f| \le W \\ 0 & \text{otherwise} \end{cases}$$

and hence

$$P_\text{n,o} = 2\int_0^W S_{N_\text{o}}(f)df = \frac{2N_0 W}{A_c^2\alpha^2} \; .$$

Since the signal component of $y_\text{o}(t)$ is $k_p m(t)$, it is clear that $P_\text{s,o} = k_p^2 P_m$. Furthermore,

$$P_\text{s,i} = \frac{A_c^2\alpha^2}{2} \quad \Rightarrow \quad \overline{\Gamma} = \frac{A_c^2\alpha^2}{2N_0W} \; .$$

Thus

$$\Gamma_\text{o} = \frac{k_p^2 P_m A_c^2\alpha^2}{2N_0W} = k_p^2 P_m \overline{\Gamma} \; = \; \beta_p^2 \; \frac{P_m}{(m_\text{max})^2} \; \overline{\Gamma} = \beta_p^2 \; P_{m_n} \; \overline{\Gamma} \; . \tag{2}$$

where $m_n(t) = m(t)/m_\text{max}$ is the normalized message signal.

Note that Γ_o grows with β_p^2, and for large enough β_p we can have Γ_o larger (or even much larger) than $\overline{\Gamma}$. However, the tradeoff of bandwidth for SNR cannot be obtained indefinitely. Eventually the small noise assumption (see ② above) is violated, and (2) no longer holds. In fact, for very large β_p, Γ_o actually decreases with β_p and eventually becomes worse than $\overline{\Gamma}$.

<u>Output SNR for FM</u>: For FM,

$$y_2(t) = k_f m(t) + n_2(t)$$

where $n_2(t) = \frac{1}{2\pi} n'_\phi(t)$.

$$S_{N_2}(f) = \frac{1}{(2\pi)^2} \; (2\pi f)^2 \; S_{N_\phi}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2\alpha^2} & \text{if } |f| \le B_\text{c}/2 \\ 0 & \text{otherwise} \end{cases}$$

and

$$y_o(t) = [y_2(t)]_\text{LPF} = k_f m(t) + n_o(t)$$

where the PSD of $N_o(t)$ is given by

$$S_{N_o}(f) = |H_\text{LPF}(f)|^2 \; S_{N_2}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2\alpha^2} & \text{if } |f| \le W \\ 0 & \text{otherwise} \end{cases} \; .$$

since $W < B_\text{c}/2$.

Hence

$$P_\text{n,o} = 2\int_0^W S_{N_o}(f)df = \frac{2N_0}{A_c^2\alpha^2}\int_0^W f^2 df = \frac{2N_0W^3}{3A_c^2\alpha^2} \; .$$

Also, it is clear that $P_\text{s,o} = k_f^2 P_m$ and $\overline{\Gamma} = A_c^2\alpha^2/(2N_0W)$. Thus

$$\Gamma_\text{o} = \frac{A_c^2\alpha^2}{2N_0W} \; \frac{3k_f^2 P_m}{W^2} = \frac{3k_f^2(m_\text{max})^2}{W^2} \; \frac{P_m}{(m_\text{max})^2} \; \overline{\Gamma} = 3\beta_f^2 \; \frac{P_m}{(m_\text{max})^2} \; \overline{\Gamma} = 3\beta_f^2 \; P_{m_n} \; \overline{\Gamma} \tag{3}$$

Again Γ_o grows with β_f^2, and for large enough β_f we can have Γ_o larger (or even much larger) than $\overline{\Gamma}$, as long as the low noise assumption of ② holds, i.e., as long as

$$\frac{\overline{\Gamma}}{2(\beta_f + 1)} > 10$$

Threshold Effect in Angle Modulation

If condition ② does not hold, the angle modulation sytem is said to be in *threshold.*

For fixed $\overline{\Gamma}$, $\beta > \frac{\overline{\Gamma}}{20} - 1 \Rightarrow$ the system is in threshold.

For fixed β, $\overline{\Gamma} < 20(\beta + 1) \Rightarrow$ the system is in threshold.

In threshold, equations (2) and (3) do not hold, and in fact, Γ_o is smaller than the right hand side of these equations. When the system is "deep" into threshold, Γ_o is worse than $\overline{\Gamma}$. So, before applying (2) or (3), we must check to make sure that the system is not in threshold.

Pre-emphasis and De-emphasis (PD) in FM

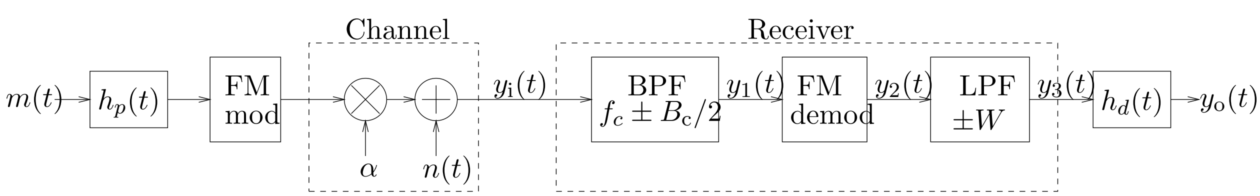
Recall that for FM,

$$y_o(t) = k_f m(t) + n_o(t)$$

where the PSD of $N_o(t)$ is given by

$$S_{N_o}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2 \alpha^2} & \text{if } |f| \le W \\ 0 & \text{otherwise} \end{cases} .$$

Thus, high frequencies in the signal (those close to W) see more noise than low frequencies at the output of the demodulator. We can exploit the non-white nature of the noise to reduce the noise power at the output via filtering. However, a filter applied to $y_o(t)$ will distort the message signal. We can get around this problem by prefiltering $m(t)$ as shown below:



The filters with impulse responses $h_p(t)$ and $h_d(t)$ are the pre-emphasis and de-emphasis filters, respectively. They are usually chosen to be first-order Butterworth low-pass and high pass filters with

$$|H_d(f)|^2 = \frac{1}{1 + \left(\frac{f}{f_0}\right)^2} \qquad |H_p(f)|^2 = 1 + \left(\frac{f}{f_0}\right)^2$$

with $h_p(t) \star h_d(t) = \delta(t)$. The cut-off frequency f_0 is chosen to be less than W.

Note that FM with PD is a general angle modulation scheme which lies somewhere in between FM and PM.

The output of the LPF (before de-emphasis) is given by:

$$y_3(t) = k_f\; m(t) \star h_p(t) + n_3(t)\,.$$

where $n_3(t)$ has the same PSD as $n_o(t)$ without PD that we studied before, i.e.,

$$S_{N_3}(f) = \begin{cases} \frac{N_0 f^2}{A_c^2\alpha^2} & \text{if } |f| \le W \\ 0 & \text{otherwise} \end{cases}\,.$$

<u>Key Point:</u> $m(t)$ "sees" both filters and is hence unaffected by PD, whereas $n(t)$ "sees" only $h_d(t)$. Hence, with PD

$$y_o(t) = k_f m(t) \star h_p(t) \star h_d(t) + n_3(t) \star h_d(t) = k_f m(t) \star \delta(t) + n_o(t) = k_f m(t) + n_o(t)$$

where $n_o(t)$ is $n_3(t)$ filtered by $h_d(t)$. Thus

$$S_{N_o}(f) = |H_d(f)|^2 S_{N_3}(f) = \begin{cases} \frac{1}{1+\left(\frac{f}{f_0}\right)^2} \frac{N_0 f^2}{A_c^2\alpha^2} & \text{if } |f| \le W \\ 0 & \text{otherwise} \end{cases}\,.$$

Hence the noise power at the output is

$$P_{\text{n,o}} = 2\int_0^W S_{N_o}(f)df = \frac{2N_0 f_0^3}{A_c^2\alpha^2}\left[\frac{W}{f_0} - \tan^{-1}\left(\frac{W}{f_0}\right)\right]$$

Recall that without PD (assuming system is not in threshold)

$$(P_{\text{n,o}})_{\text{FM}} = \frac{2N_0 W^3}{3A_c^2\alpha^2}\,.$$

and $P_{\text{s,o}}$ is the same with and without PD. Thus

$$\Gamma_{o,\text{FM-PD}} = \Gamma_{o,\text{FM}}\;\frac{(P_{\text{n,o}})_{\text{FM}}}{(P_{\text{n,o}})_{\text{FM-PD}}} = \Gamma_{o,\text{FM}}\;\frac{\left(\frac{W}{f_0}\right)^3}{3\left[\frac{W}{f_0} - \tan^{-1}\left(\frac{W}{f_0}\right)\right]}$$

It is easy to show that $\Gamma_{o,\text{FM-PD}} > \Gamma_{o,\text{FM}}$ if $f_0 < W$.

<u>Example</u> For FM broadcasting, $B = 15$ kHz, $\beta = 5$ and $f_0 = 2\pi \times 2100$ rad/s. If the average-to-peak power ratio is 0.5 and $\overline{\Gamma} = 30$ dB, we can show that $\Gamma_{o,\text{FM-PD}}$ =45.74 dB (after cheking that system is not in threshold) and that $\Gamma_{o,\text{FM-PD}}$ =59.1 dB. Thus we get almost a 30 dB improvement in SNR over DSB-SC and even more over conventional AM. This is why FM radio sounds so much better than AM radio!