ECE 359 **Spring 2003**

Handout # 18 April 22, 2003

DIGITAL COMMUNICATIONS

- There are various sources of digital information – they are naturally digital (e.g. data, text, etc.), or analog sources converted to digital via sampling and quantization.

- Any sequence of digital information can be converted into a *binary* sequence, but we may choose to send the information on the channel in non-binary form.

- Digital communication can be done in *baseband* or in *passband* (via carrier modulation)

  ◇ Baseband examples: computer to peripheral communications, magnetic recording, etc.

  ◇ Passband examples: wireless communications, wireline communications with multiplexing.

Signal Sets for Binary Digital Communications

- We begin by restricting attention to binary (0's and 1's) information sequences.

- Since we can only transmit continuous (analog) signals on an electrical communications channel, we need to convert the bit stream into an analog waveform. This process is sometimes called *digital modulation.*

- We assign a (continuous) waveform to each of 0 and 1; $s_0(t)$ is assigned to 0 and $s_1(t)$ is assigned to 1. To send a sequence of bits on the channel we send the corresponding sequence of waveforms.

- Suppose the data is to be transmitted at the rate of one bit every T seconds (T is call the bit-rate). Then we do the following:

  1. We limit $s_0(t)$ and $s_1(t)$ to $[0, T]$, i.e., we pick the waveforms such that
  $$s_i(t) = 0 \text{ for } t < 0 \text{ and } t > T, \quad i = 0, 1 \;.$$
  (In practice, $s_0(t)$ and $s_1(t)$ may span several multiples of T since such waveforms have better bandwidth properties.)
  2. We pick the waveforms $s_0(t)$ and $s_1(t)$ to be either baseband (frequency content around 0) or passband (frequency content around some center frequency f_c)
  3. To send the bit sequence $b_0\ b_1\ \cdots\ b_{k-1}$ we associate bits with waveforms in the following way:
  $$\begin{array}{lcl} b_0 & \longrightarrow & s_{b_0}(t) \\ b_1 & \longrightarrow & s_{b_1}(t-T) \\ b_2 & \longrightarrow & s_{b_2}(t-2T) \\ \vdots & & \vdots \\ b_{k-1} & \longrightarrow & s_{b_{k-1}}(t-(k-1)T) \end{array}$$
  Then the transmitted waveform corresponding to the bit sequence is:
  $$v(t) = \sum_{j=0}^{k-1} s_{b_j}(t - jT) \;.$$

- **Some Definitions**

  ◇ *Unit Pulse of Duration T*

$$p_T(t) = \begin{cases} 1 & \text{if } 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$$

  ◇ *Signal Energies:* For $i = 0, 1$,

$$\mathcal{E}_i = \int_{-\infty}^{\infty} [s_i(t)]^2 dt = \int_0^T [s_i(t)]^2 dt$$

  ◇ *Types of Signal Sets* (examples done in class):

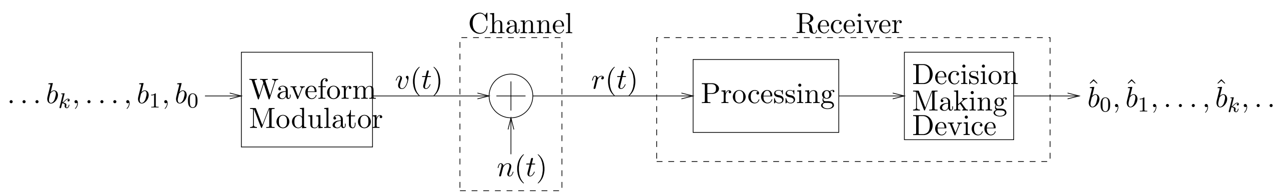
  1. On-Off Keying: $s_1(t) \neq 0$, $s_0(t) = 0 \Rightarrow \mathcal{E}_1 > 0$ and $\mathcal{E}_0 = 0$.
  2. Antipodal: $s_1(t) = s(t)$, $s_0(t) = -s(t) \Rightarrow \mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}$.
  3. Orthogonal: $s_0(t)$ and $s_1(t)$ satisfy (a) $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}$ and (b) $\int_0^T s_0(t)s_1(t)dt = 0$

- **Transmission on AWGN Channel**

  ◇ *Goal:* To recover transmitted bits at receiver with as few errors as possible.

  ◇ Without noise, there are many ways to recover information (we gave examples in class).

  ◇ System for binary communications on AWGN Channel:



  where $v(t) = \sum_j s_{b_j}(t - jT)$

- **Optimum Detection of One Bit**

  ◇ Assume that bits are processed separately. This okay since

  ○ bit waveforms do not overlap

  ○ bits are assumed to be independent

  ○ noise is white

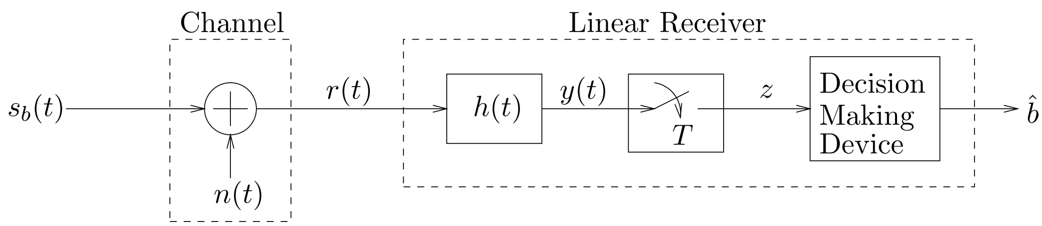
  ◇ Without loss of generality, we focus on the reception of first bit b_0, and we drop the subscript "0" for covenience. The relevant time interval for detection of this bit is the interval $[0, T]$, and the received signal on this interval is given by:

$$\boxed{r(t) = s_b(t) + n(t)\ , \quad 0 \le t \le T}$$

  ◇ Detection/Estimation Problem: Find the best estimate $\hat{b}$ of b at the receiver based on $r(t), 0 \le t \le T$.

  ◇ We will solve this optimization problem for a general class of *linear* receivers. It can be shown that there is no loss of optimality in restricting to linear receivers.

- **General Linear Receiver Structure**



Goal: To design $h(t)$ and the decision device to minimize the probability of bit error $\mathrm{P_e} = \mathrm{P}\{\hat{b} \neq b\}$.

- **Optimizing Decision Device**

Decision is based on $z = y(T)$. Now

$$y(t) = r(t) \star h(t) = \int_{-\infty}^{\infty} h(\xi) r(t-\xi) d\xi \quad \Rightarrow \quad z = y(T) = \int_{-\infty}^{\infty} h(\xi) r(T-\xi) d\xi \, .$$

We can express z as $z = \mu_b + x$ where

$$\mu_b = \int_{-\infty}^{\infty} h(\xi) s_b(T-\xi) d\xi \ \text{ and } \ x = \int_{-\infty}^{\infty} h(\xi) n(T-\xi) d\xi \, .$$

The corresponding equation relating the random variables Z and X is given by

$$\boxed{Z = \mu_b + X}$$

Note that since the filter is linear, X is a Gaussian random variable with mean $\mathrm{E}[X] = 0$ and variance

$$\mathrm{var}[X] = \mathrm{E}[X^2] - 0 = \mathrm{E}\left[\left(\int_{-\infty}^{\infty} h(\xi) N(T-\xi) d\xi\right)^2\right] = \cdots = \frac{N_0}{2} \int_{-\infty}^{\infty} [h(\xi)]^2 d\xi \ \stackrel{\text{def}}{=} \ \sigma^2 \, .$$

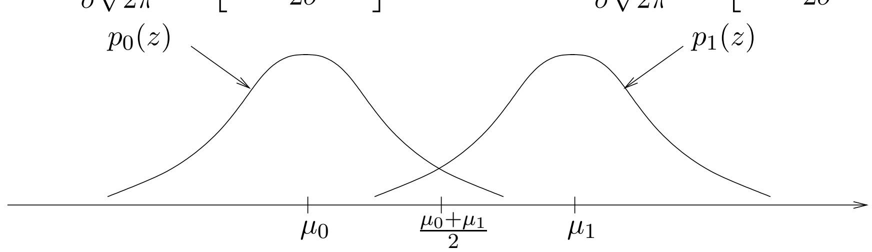
- **Hypothesis Testing**

$$H_0 : \text{bit `0' was sent} \quad \Rightarrow \quad Z = \mu_0 + X$$
$$H_1 : \text{bit `1' was sent} \quad \Rightarrow \quad Z = \mu_1 + X$$

Key Point: The pdf of Z depends on the value of b. We can exploit this information to form the estimate $\hat{b}$.

Under hypothesis H_0, the observation $Z \sim \mathcal{N}(\mu_0, \sigma^2)$; and under H_1, $Z \sim \mathcal{N}(\mu_1, \sigma^2)$. If we denote the pdf's by $p_0(z)$ and $p_1(z)$, respectively, then

$$p_0(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(z-\mu_0)^2}{2\sigma^2}\right] \ , \text{ and } \ p_1(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(z-\mu_1)^2}{2\sigma^2}\right]$$



- **Maximum Likelihood Decision Making:** The pdfs $p_0(z)$ and $p_1(z)$ are likelihood functions for the random variable Z. The maximum likelihood decision rule is described by

$$\text{pick '1' if } p_1(z) \geq p_0(z)$$
$$\text{pick '0' if } p_1(z) < p_0(z)$$

This decision rule can be rewritten as:

$$\text{likelihood ratio} \longrightarrow \frac{p_1(z)}{p_0(z)} \overset{1}{\underset{0}{\gtrless}} 1 \longleftarrow \text{threshold}$$

and is hence also called a likelihood ratio test (LRT).

Recall that our goal is to minimize $\mathrm{P_e} = \mathrm{P}\{\hat{b} \neq b\}$. It is not difficult to show that the maximum likelihood decision rule also minimizes $\mathrm{P_e}$, if the bits '0' and '1' are equally likely (see HW 10).

We can simplify the optimum decision rule as follows:

$$\frac{p_1(z)}{p_0(z)} \overset{1}{\underset{0}{\gtrless}} 1 \quad \Leftrightarrow \quad \frac{\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(z-\mu_1)^2}{2\sigma^2}\right]}{\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(z-\mu_0)^2}{2\sigma^2}\right]} \overset{1}{\underset{0}{\gtrless}} 1 \quad \Leftrightarrow \quad \cdots \quad \Leftrightarrow \quad (\mu_1 - \mu_0)\, z \overset{1}{\underset{0}{\gtrless}} \frac{\mu_1^2 - \mu_0^2}{2} \,.$$

If we assume that $\mu_1 > \mu_0$, then we can divide both sides of the inequality by $(\mu_1 - \mu_0)$ to get the optimum decision rule as

$$\boxed{z \overset{1}{\underset{0}{\gtrless}} \lambda = \frac{\mu_0 + \mu_1}{2}} \tag{1}$$

If $\mu_1 < \mu_0$, the inequalities simply get reversed.

- **Probability of Error:**

There are two types of error:

$$\mathrm{P_{e,0}} = \mathrm{P}(\{\hat{b} = 1\} | \{b = 0\} \,, \quad \text{and } \mathrm{P_{e,1}} = \mathrm{P}(\{\hat{b} = 0\} | \{b = 1\}) \,.$$

For equally likely bits, using Bayes rule, we get

$$\mathrm{P_e} = \frac{1}{2}\mathrm{P_{e,0}} + \frac{1}{2}\mathrm{P_{e,1}} \,.$$

For the optimum decision rule of (1) (assuming $\mu_1 > \mu_0$) we can easily show that

$$\mathrm{P_{e,0}} = \mathrm{P}(\{Z > \lambda\} | \{b = 0\}) = \mathrm{P_0}\{Z > \lambda\} = \cdots = Q\left(\frac{\mu_1 - \mu_0}{2\sigma}\right)$$

and that

$$\mathrm{P_{e,1}} = \mathrm{P}(\{Z < \lambda\} | \{b = 1\}) = \mathrm{P_1}\{Z < \lambda\} = \cdots = Q\left(\frac{\mu_1 - \mu_0}{2\sigma}\right) \,.$$

That is $\mathrm{P_{e,0}} = \mathrm{P_{e,1}}$. Thus

$$\boxed{\mathrm{P_e} = Q\left(\frac{\mu_1 - \mu_0}{2\sigma}\right)}$$