DIGITAL COMMUNICATIONS

- There are various sources of digital information – they are naturally digital (e.g. data, text, etc.), or analog sources converted to digital via sampling and quantization.

- Any sequence of digital information can be converted into a binary sequence, but we may choose to send the information on the channel in non-binary form.

- Digital communication can be done in baseband or in passband (via carrier modulation)
  - Baseband examples: computer to peripheral communications, magnetic recording, etc.
  - Passband examples: wireless communications, wireline communications with multiplexing.

**Signal Sets for Binary Digital Communications**

- We begin by restricting attention to binary (0’s and 1’s) information sequences.

- Since we can only transmit continuous (analog) signals on an electrical communications channel, we need to convert the bit stream into an analog waveform. This process is sometimes called digital modulation.

- We assign a (continuous) waveform to each of 0 and 1; $s_0(t)$ is assigned to 0 and $s_1(t)$ is assigned to 1. To send a sequence of bits on the channel we send the corresponding sequence of waveforms.

- Suppose the data is to be transmitted at the rate of one bit every $T$ seconds ($T$ is call the bit-rate). Then we do the following:

  1. We limit $s_0(t)$ and $s_1(t)$ to $[0, T]$, i.e., we pick the waveforms such that

     \[ s_i(t) = 0 \text{ for } t < 0 \text{ and } t > T, \quad i = 0, 1. \]

     (In practice, $s_0(t)$ and $s_1(t)$ may span several multiples of $T$ since such waveforms have better bandwidth properties.)

  2. We pick the waveforms $s_0(t)$ and $s_1(t)$ to be either baseband (frequency content around 0) or passband (frequency content around some center frequency $f_c$)

  3. To send the bit sequence $b_0 \ b_1 \ \cdots \ b_{k-1}$ we associate bits with waveforms in the following way:

     \[
     \begin{align*}
     b_0 & \quad \longrightarrow \quad s_{b_0}(t) \\
     b_1 & \quad \longrightarrow \quad s_{b_1}(t - T) \\
     b_2 & \quad \longrightarrow \quad s_{b_2}(t - 2T) \\
     \vdots & \quad \vdots \\
     b_{k-1} & \quad \longrightarrow \quad s_{b_{k-1}}(t - (k - 1)T)
     \end{align*}
     \]

     Then the transmitted waveform corresponding to the bit sequence is:

     \[ v(t) = \sum_{j=0}^{k-1} s_{b_j}(t - jT). \]
• Some Definitions

- **Unit Pulse of Duration** $T$
  \[ p_T(t) = \begin{cases} 
  1 & \text{if } 0 \leq t \leq T \\
  0 & \text{otherwise} 
  \end{cases} \]

- **Signal Energies**: For $i = 0, 1$,
  \[ E_i = \int_{-\infty}^{\infty} [s_i(t)]^2 dt = \int_0^T [s_i(t)]^2 dt \]

- **Types of Signal Sets** (examples done in class):
  1. On-Off Keying: $s_1(t) \neq 0$, $s_0(t) = 0 \Rightarrow E_1 > 0$ and $E_0 = 0$.
  2. Antipodal: $s_1(t) = s(t)$, $s_0(t) = -s(t) \Rightarrow E_0 = E_1 = E$.
  3. Orthogonal: $s_0(t)$ and $s_1(t)$ satisfy (a) $E_0 = E_1 = E$ and (b) $\int_0^T s_0(t)s_1(t)dt = 0$

• Transmission on AWGN Channel

- **Goal**: To recover transmitted bits at receiver with as few errors as possible.
- Without noise, there are many ways to recover information (we gave examples in class).
- System for binary communications on AWGN Channel:

- Optimum Detection of One Bit

  - Assume that bits are processed separately. This okay since
    - bit waveforms do not overlap
    - bits are assumed to be independent
    - noise is white
  
  - Without loss of generality, we focus on the reception of first bit $b_0$, and we drop the subscript “0” for convenience. The relevant time interval for detection of this bit is the interval $[0, T]$, and the received signal on this interval is given by:
    \[ r(t) = s_0(t) + n(t), \quad 0 \leq t \leq T \]

  - Detection/Estimation Problem: Find the best estimate $\hat{b}$ of $b$ at the receiver based on $r(t), 0 \leq t \leq T$.
  - We will solve this optimization problem for a general class of linear receivers. It can be shown that there is no loss of optimality in restricting to linear receivers.
General Linear Receiver Structure

\[ s_b(t) \rightarrow n(t) \rightarrow r(t) \rightarrow h(t) y(t) \rightarrow z \rightarrow \hat{b} \]

**Goal:** To design \( h(t) \) and the decision device to minimize the probability of bit error \( P_e = P\{\hat{b} \neq b\} \).

**Optimizing Decision Device**

Decision is based on \( z = y(T) \). Now

\[ y(t) = r(t) * h(t) = \int_{-\infty}^{\infty} h(\xi) r(t - \xi) d\xi \quad \Rightarrow \quad z = y(T) = \int_{-\infty}^{\infty} h(\xi) r(T - \xi) d\xi. \]

We can express \( z \) as \( z = \mu_b + x \) where

\[ \mu_b = \int_{-\infty}^{\infty} h(\xi) s_b(T - \xi) d\xi \quad \text{and} \quad x = \int_{-\infty}^{\infty} h(\xi) n(T - \xi) d\xi. \]

The corresponding equation relating the random variables \( Z \) and \( X \) is given by

\[ Z = \mu_b + X \]

Note that since the filter is linear, \( X \) is a Gaussian random variable with mean \( E[X] = 0 \) and variance

\[ \text{var}[X] = E[X^2] - 0 = E \left[ \left( \int_{-\infty}^{\infty} h(\xi) N(T - \xi) d\xi \right)^2 \right] = \cdots = \frac{N_0}{2} \int_{-\infty}^{\infty} [h(\xi)]^2 d\xi \overset{\text{def}}{=} \sigma^2. \]

**Hypothesis Testing**

\[ H_0 : \text{bit '0' was sent} \quad \Rightarrow \quad Z = \mu_0 + X \]
\[ H_1 : \text{bit '1' was sent} \quad \Rightarrow \quad Z = \mu_1 + X \]

**Key Point:** The pdf of \( Z \) depends on the value of \( b \). We can exploit this information to form the estimate \( \hat{b} \).

Under hypothesis \( H_0 \), the observation \( Z \sim N(\mu_0, \sigma^2) \); and under \( H_1 \), \( Z \sim N(\mu_1, \sigma^2) \). If we denote the pdf’s by \( p_0(z) \) and \( p_1(z) \), respectively, then

\[ p_0(z) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(z - \mu_0)^2}{2\sigma^2} \right], \quad \text{and} \quad p_1(z) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(z - \mu_1)^2}{2\sigma^2} \right] \]
**Maximum Likelihood Decision Making:** The pdfs $p_0(z)$ and $p_1(z)$ are likelihood functions for the random variable $Z$. The maximum likelihood decision rule is described by

- pick ‘1’ if $p_1(z) \geq p_0(z)$
- pick ‘0’ if $p_1(z) < p_0(z)$

This decision rule can be rewritten as:

\[
\text{likelihood ratio} \rightarrow \frac{p_1(z)}{p_0(z)} \gtrless 0 \quad \text{threshold}
\]

and is hence also called a likelihood ratio test (LRT).

Recall that our goal is to minimize $P_e = P\{\hat{b} \neq b\}$. It is not difficult to show that the maximum likelihood decision rule also minimizes $P_e$, if the bits ‘0’ and ‘1’ are equally likely (see HW 10).

We can simplify the optimum decision rule as follows:

\[
p_1(z) \gtrless 0 \quad \text{if } \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{(z-\mu_1)^2}{2\sigma^2} \right] \gtrless 0 \quad \text{and } \quad \left( \mu_1 - \mu_0 \right) z \gtrless 0 \quad \frac{\mu_1^2 - \mu_0^2}{2}.
\]

If we assume that $\mu_1 > \mu_0$, then we can divide both sides of the inequality by $(\mu_1 - \mu_0)$ to get the optimum decision rule as

\[
z > 0 \quad \lambda = \frac{\mu_0 + \mu_1}{2} \quad (1)
\]

If $\mu_1 < \mu_0$, the inequalities simply get reversed.

**Probability of Error:**

There are two types of error:

\[
P_{e,0} = P(\{\hat{b} = 1\}|\{b = 0\}) \quad \text{and} \quad P_{e,1} = P(\{\hat{b} = 0\}|\{b = 1\}).
\]

For equally likely bits, using Bayes rule, we get

\[
P_e = \frac{1}{2} P_{e,0} + \frac{1}{2} P_{e,1}.
\]

For the optimum decision rule of (1) (assuming $\mu_1 > \mu_0$) we can easily show that

\[
P_{e,0} = P(\{Z > \lambda\}|\{b = 0\}) = P_0\{Z > \lambda\} = \cdots = Q\left( \frac{\mu_1 - \mu_0}{2\sigma} \right)
\]

and that

\[
P_{e,1} = P(\{Z < \lambda\}|\{b = 1\}) = P_1\{Z < \lambda\} = \cdots = Q\left( \frac{\mu_1 - \mu_0}{2\sigma} \right).
\]

That is $P_{e,0} = P_{e,1}$. Thus

\[
P_e = Q\left( \frac{\mu_1 - \mu_0}{2\sigma} \right)
\]