MATH REVIEW

COMPLEX VARIABLES

A complex number \( z \) is expressed as
\[
z = x + jy,
\]
where \( x \) and \( y \) are real numbers, and \( j \) is the square root of \(-1\) (i.e., \( j^2 = -1 \) and \( 1/j = -j \)).

The real part of \( z \) is denoted by \( \text{Re}(z) \) and the imaginary part by \( \text{Im}(z) \). That is,
\[
x = \text{Re}(z) \quad \text{and} \quad y = \text{Im}(z)
\]

The complex conjugate of \( z \) is denoted by \( \overline{z} \), and \( \overline{z} = x - jy \). Clearly,
\[
\text{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \text{Im}(z) = \frac{z - \overline{z}}{2j}
\]

The magnitude of \( z \) is denoted by \( |z| \), and the angle (or phase) of \( z \) is denoted by \( \angle z \).
\[
|z| = \sqrt{x^2 + y^2} = \sqrt{[\text{Re}(z)]^2 + [\text{Im}(z)]^2} \quad \angle z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)
\]

With this understanding, \( z \) can be expressed in terms of \( |z| \) and \( \angle z \) in the following way.
\[
z = |z| \left(\cos(\angle z) + j\sin(\angle z)\right)
\]

**Exercise:** Prove that Equation (1) is true.

**Example:** \( z = 4 + 7j \)
\[
\text{Re}(z) = 4 \quad \text{Im}(z) = 7
\]
\[
|z| = \sqrt{4^2 + 7^2} = 8.062 \quad \angle z = \tan^{-1}(7/4) = 1.05 \text{ radians}
\]

**Complex Exponential Function:** For any real number \( a \), the complex exponential function of \( a \) is defined by
\[
e^{ja} = \cos a + j\sin a
\]

Plugging the above in Equation (1), we get
\[
z = |z| e^{j\angle z}
\]

This is called the polar form of the complex number \( z \).

The complex exponential function has properties similar to the real exponential function that you are very familiar with. For example,
\[
e^{j(a+b)} = e^{ja} e^{jb} \quad e^{j(a-b)} = e^{ja} e^{-jb} \quad \int e^{ja} da = \frac{e^{ja}}{j} \quad \frac{d}{da} e^{ja} = j e^{ja}
\]
All other operations using the complex exponential function such as integration by parts, chain rule, etc. work exactly like those on the real exponential function with the understanding that we need to keep track of the “j” in the expressions.

**Operations on Complex Numbers**

Let \( z_1 \) and \( z_2 \) be any two complex numbers. Then

**Addition and Subtraction:**

\[
z_1 + z_2 = \text{Re}(z_1) + \text{Re}(z_2) + j[\text{Im}(z_1) + \text{Im}(z_2)]
\]

\[
z_1 - z_2 = \text{Re}(z_1) - \text{Re}(z_2) + j[\text{Im}(z_1) - \text{Im}(z_2)]
\]

**Multiplication and Division:**

Examples:

\[
(2 + 3j)(4 + j) = 8 + 2j + 12j + 3j^2 = 5 + 14j \quad \text{(remember } j^2 = -1)\]

\[
\frac{2 + 3j}{4 + j} = \frac{(2 + 3j)(4 - j)}{(4 + j)(4 - j)} = \frac{11 + 10j}{17}
\]

Multiplication and Division are easier in polar form:

\[
z_1z_2 = |z_1||z_2|e^{j(\theta_1 + \theta_2)}
\]

\[
\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}e^{j(\theta_1 - \theta_2)}
\]

**Exponentiation:** (also easier in polar form)

\[
(1 + j\sqrt{3})^4 = \left(2e^{j\pi/3}\right)^4 = 2^4e^{4j\pi/3} = 16(\cos(4\pi/3) + j\sin(4\pi/3)) = -8 - 8j\sqrt{3}
\]

**Roots:** (also easier in polar form)

Example: Find all the cube roots of 8

\[
8^{1/3} = (8e^{0j})^{1/3} = 8^{1/3}e^{j(0 + 2\pi k/3)}, k = 0, 1, 2 = 2, 2e^{j2\pi/3}, 2e^{j4\pi/3}
\]

**TRIGONOMETRIC IDENTITIES**

\[
\cos(-x) = \cos(x) \quad \sin(-x) = -\sin(x) \quad \cos^2(x) + \sin^2(x) = 1
\]
\[
\cos(x+y) = \cos x \cos y - \sin x \sin y \quad \sin(x+y) = \sin x \cos y + \sin y \cos x
\]

\[
\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x \quad \sin(2x) = 2\sin x \cos x
\]

\[
\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}
\]

\[
\cos x \cos y = \frac{\cos(x+y) + \cos(x-y)}{2} \quad \sin x + \sin y = 2 \sin \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)
\]

\[
\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2} \quad \sin x - \sin y = 2 \cos \left( \frac{x+y}{2} \right) \sin \left( \frac{x-y}{2} \right)
\]

\[
\sin x \cos y = \frac{\sin(x+y) + \sin(x-y)}{2} \quad \cos x + \cos y = 2 \cos \left( \frac{x+y}{2} \right) \cos \left( \frac{x-y}{2} \right)
\]

\[
\sin y \cos x = \frac{\sin(x+y) - \sin(x-y)}{2} \quad \cos x - \cos y = -2 \sin \left( \frac{x+y}{2} \right) \sin \left( \frac{x-y}{2} \right)
\]

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2j}
\]

\[
A \cos(x) + B \sin(x) = \sqrt{A^2 + B^2} \cos(x - \tan^{-1}(B/A))
\]

**LOGARITHMS**

*Definition:* The log function is the inverse of the exponential function.

\[
a^x = b \iff \log_a b = x
\]

where \(a\) is the base of the log function. Commonly used bases are 10 and \(e\) (for natural log or the ln function). If a base is not specified it is usually taken to be 10.

*Properties:*

End point conditions: \(\log_a a = 1\) \quad \(\log_a(0) = -\infty \quad (a > 1)\) \quad \(\log_a(1) = 0\) \quad \(\log_a\) (negative num) = undefined

Base conversion: \(\log_a b = \frac{\ln(b)}{\ln(a)} = \frac{\log(b)}{\log(a)} = \frac{\log_e b}{\log_e a}\)

Multiplication and Division: \(\log_a(xy) = \log_a x + \log_a y\) \quad \(\log_a(x/y) = \log_a x - \log_a y\)
Exponents in logs: \( \log_a(b^r) = r \log_a b \)

Inverse operations: \( a^{\log_a x} = x \quad \log_a a^x = x \quad 2^x = e^{\ln(2^x)} = e^x \ln 2, \text{ etc.} \)

**SEQUENCES AND SERIES**

**Taylor Series:** \( f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \), where \( f^{(n)}(a) \) is the \( n \)-th derivative of \( f \) evaluated at \( a \).

**Examples**

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad \text{for} \ |x| < 1
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots
\]

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

**Other Infinite Series**

Euler’s Definition of \( e \):

\[
\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x
\]

Finite Geometric Series:

\[
a \sum_{i=0}^{n} r^i = a + ar + ar^2 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1 - r}
\]

Infinite Geometric Series:

\[
a \sum_{i=0}^{\infty} r^i = a + ar + ar^2 + \cdots = \frac{a}{1 - r}, \text{ if} \ |r| < 1
\]

Sum of Numbers:

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]

Sum of Squares:

\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

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