ECE 359 **Spring 2003**

Handout # 20 April 24, 2003

DIGITAL COMMUNICATIONS (cont.)

- For $\mu_1 > \mu_0$, $\mathrm{P_e}$ is given by

$$\mathrm{P_e} = Q\left(\frac{\mu_1 - \mu_0}{2\sigma}\right) .$$

If $\mu_1 < \mu_0$, it easy to show by a symmetry argument (swapping μ_0 and μ_1) that

$$\mathrm{P_e} = Q\left(\frac{\mu_0 - \mu_1}{2\sigma}\right) .$$

Thus

$$\boxed{\mathrm{P_e} = Q\left(\frac{|\mu_1 - \mu_0|}{2\sigma}\right)} \tag{1}$$

- **Minimizing $\mathbf{P_e}$ and Maximizing SNR:**

Since $Q(\cdot)$ is a monotonically decreasing function, the smallest value of $\mathrm{P_e}$ is obtained when $\frac{|\mu_1-\mu_0|}{2\sigma}$ is maximized, or equivalently if the following quantity ρ is maximized.

$$\rho = \frac{|\mu_1 - \mu_0|^2}{\sigma^2} .$$

Note that ρ is a measure of the *signal-to-noise* ratio at the output of the filter.

Recall that

$$\mu_i = \int_{-\infty}^{\infty} h(\xi) s_i(T-\xi) d\xi \, , i = 0, 1 \, , \quad \text{and} \quad \sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} [h(\xi)]^2 d\xi$$

Thus

$$\rho = \frac{\left[\int_{-\infty}^{\infty} h(\xi)[s_1(T-\xi) - s_0(T-\xi)]d\xi\right]^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}[h(\xi)]^2 d\xi} \tag{2}$$

- **Optimizing the filter** $h(\cdot)$

Our goal is to find $h(t)$ that maximimzes ρ. This maximization is facilitated by the Cauchy-Schwarz inequality.

Cauchy-Schwarz (C-S) Inequality: For finite energy signals $x_1(t)$ and $x_2(t)$

$$\left[\int_{-\infty}^{\infty} x_1(t) x_2(t) dt\right]^2 \quad \leq \quad \int_{-\infty}^{\infty} [x_1(t)]^2 \, dt \int_{-\infty}^{\infty} [x_2(t)]^2 \, dt$$

with equality iff $x_2(t) = \alpha \, x_1(t)$ for some real number α.

Applying the C-S inequality to $h(\xi)$ and $[s_1(T-\xi) - s_0(T-\xi)]$ we get

$$\left[\int_{-\infty}^{\infty} h(\xi)[s_1(T-\xi) - s_0(T-\xi)]d\xi\right]^2 \quad \leq \quad \int_{-\infty}^{\infty} [h(\xi)]^2 \, d\xi \int_{-\infty}^{\infty} [s_1(T-\xi) - s_0(T-\xi)]^2 \, d\xi \, .$$

Dividing both sides by $\frac{N_0}{2} \int_{-\infty}^{\infty} [h(\xi)]^2 \, d\xi$, we get the following inequality for ρ of (2).

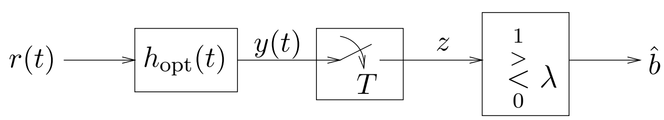
$$\rho = \frac{\left[\int_{-\infty}^{\infty} h(\xi)[s_1(T-\xi) - s_0(T-\xi)]d\xi\right]^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} [h(\xi)]^2 d\xi} \quad \leq \quad \frac{2}{N_0} \int_{-\infty}^{\infty} [s_1(T-\xi) - s_0(T-\xi)]^2 \, d\xi$$

with equality iff $h(\xi) = \alpha[s_1(T-\xi) - s_0(T-\xi)]$. Without loss of generality we can set $\alpha = 1$, since scaling the filter does not affect ρ. Thus ρ is maximized by choosing

$$\boxed{h(t) = h_{\text{opt}}(t) \; = \; s_1(T-t) - s_0(T-t)}$$

This filter $h_{\text{opt}}(t)$ is also called the *matched filter*. Note that $h_{\text{opt}}(t)$ is obtained by "flipping" the difference signal $[s_1(t) - s_0(t)]$ in the interval $[0, T]$ around $t = T/2$. Thus $h_{\text{opt}}(t) = 0$ for $t < 0$ and $t > T$. This means that the optimum filter is *causal*.

We have now optimized both the decision device and the filter, and the optimum linear receiver is given by:



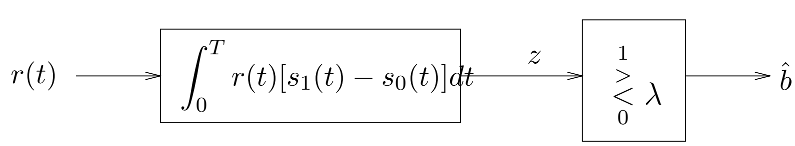
- **Optimum Receiver as a Correlation Receiver**

For the optimum linear receiver shown above

$$z = y(T) = \int_{-\infty}^{\infty} r(T-\xi) h_{\text{opt}}(\xi) d\xi = \int_{0}^{T} r(T-\xi) h_{\text{opt}}(\xi) d\xi$$

$$= \int_0^T r(T-\xi)[s_1(T-\xi) - s_0(T-\xi)]d\xi \quad \stackrel{T-\xi=t}{=} \quad \int_0^T r(t)[s_1(t) - s_0(t)]dt$$

Thus an alternative way to construct the optimum receiver is to produce z by correlating $r(t)$ with the difference $s_1(t) - s_0(t)$. This correlator implementation is shown below.



- **Performance of the Optimum Receiver**

For the optimum receiver

$$\mu_{i,\text{opt}} = \int_0^T s_i(t)[s_1(t) - s_0(t)] \, dt \, , \quad i = 0, 1$$

and hence

$$\mu_{1,\text{opt}} - \mu_{0,\text{opt}} = \int_0^T [s_1(t) - s_0(t)]^2 \, dt = d > 0 \, .$$

where d is the "squared distance" between the signals $s_1(t)$ and $s_0(t)$ and is defined by

$$d \stackrel{\text{def}}{=} \int_0^T [s_1(t) - s_0(t)]^2 \, dt$$

Since $\mu_{1,\text{opt}} - \mu_{0,\text{opt}} > 0$, the ML decision rule for the optimum receiver is

$$\boxed{z \underset{0}{\overset{1}{\gtrless}} \lambda_{\text{opt}}}$$

where

$$\lambda_{\text{opt}} = \frac{\mu_{0,\text{opt}} + \mu_{1,\text{opt}}}{2} = \frac{1}{2}\int_0^T [s_1(t) + s_0(t)]\,[s_1(t) - s_0(t)] \, dt = \frac{1}{2}\int_0^T [s_1^2(t) - s_0^2(t)] \, dt = \frac{\mathcal{E}_1 - \mathcal{E}_0}{2}$$

Also, the error probability for the optimum receiver is

$$\mathrm{P}_{\text{e,opt}} = Q\left(\frac{\mu_{1,\text{opt}} - \mu_{0,\text{opt}}}{2\sigma_{\text{opt}}}\right) = Q\left(\frac{d}{2\sigma_{\text{opt}}}\right)$$

where

$$\sigma_{\text{opt}}^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} [h_{\text{opt}}(\xi)]^2 \, d\xi = \cdots = \frac{N_0}{2}[\mu_{1,\text{opt}} - \mu_{0,\text{opt}}] = \frac{N_0}{2} \, d \, .$$

Thus

$$\boxed{\mathrm{P}_{\text{e,opt}} = Q\left(\sqrt{\frac{d}{2N_0}}\right)}$$

<u>*Special Cases*</u>

In the following, we use $\bar{\mathcal{E}}$ to denote the average signal energy $\frac{\mathcal{E}_0 + \mathcal{E}_1}{2}$.

- *On-Off Keying:* $s_1(t) \neq 0$, $s_0(t) = 0 \Rightarrow \mathcal{E}_1 > 0$, $\mathcal{E}_0 = 0$, $\bar{\mathcal{E}} = \mathcal{E}_1/2$.

$$d = \int_0^T [s_1(t)]^2 \, dt = \mathcal{E}_1 \, , \;\; \lambda_{\text{opt}} = \frac{\mathcal{E}_1}{2} \, , \;\; \text{and } \mathrm{P}_{\text{e,opt}} = Q\left(\sqrt{\frac{\mathcal{E}_1}{2N_0}}\right) = Q\left(\sqrt{\frac{\bar{\mathcal{E}}}{N_0}}\right)$$

- *Antipodal Signaling:* $s_1(t) = s(t)$, $s_0(t) = -s(t) \Rightarrow \mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E} = \bar{\mathcal{E}}$.

$$d = \int_0^T [2s(t)]^2 \, dt = 4\mathcal{E} \, , \;\; \lambda_{\text{opt}} = \frac{\mathcal{E}_1 - \mathcal{E}_0}{2} = 0 \, , \;\; \text{and } \mathrm{P}_{\text{e,opt}} = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right) = Q\left(\sqrt{\frac{2\bar{\mathcal{E}}}{N_0}}\right)$$

∘ *Orthogonal:* $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E} = \bar{\mathcal{E}}$ and $\int_0^T s_0(t)s_1(t)dt = 0$

$$d = \mathcal{E}_0 + \mathcal{E}_1 - 2\int_0^T s_0(t)s_1(t)\, dt = 2\mathcal{E}\ ,\ \lambda_{\rm opt} = \frac{\mathcal{E}_1 - \mathcal{E}_0}{2} = 0\ ,$$

and

$$\mathrm{P}_{\rm e,opt} = Q\left(\sqrt{\frac{\mathcal{E}}{N_0}}\right) = Q\left(\sqrt{\frac{\bar{\mathcal{E}}}{N_0}}\right)$$

- **Proposition** (Optimum Signal Set) Given constraints on the signal energies $\mathcal{E}_0 \leq \mathcal{E}$ and $\mathcal{E}_1 \leq \mathcal{E}$, the best signal set $\{s_0(t), s_1(t)\}$ is the *antipodal* signal set with $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}$.

  Proof The set $\{s_0(t), s_1(t)\}$ that maximizes d, minimizes $\mathrm{P}_{\rm e,opt}$. Now,

$$d = \int_0^T [s_1(t) - s_0(t)]^2\, dt = \mathcal{E}_0 + \mathcal{E}_1 - 2\int_0^T s_0(t)s_1(t)\, dt$$

  By Cauchy-Schwarz inequality

$$\int_0^T s_0(t)s_1(t)\, dt \ \geq\ -\sqrt{\mathcal{E}_0\mathcal{E}_1}$$

  Thus

$$d \leq \mathcal{E}_0 + \mathcal{E}_1 + 2\sqrt{\mathcal{E}_0\mathcal{E}_1} \leq \mathcal{E} + \mathcal{E} + 2\sqrt{\mathcal{E}^2} = 4\mathcal{E}$$

  But the bound of $4\mathcal{E}$ is achieved when we pick antipodal signals with $\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}$. Hence, antipodal signals are optimum.

- **Binary Coherent Communication:** For binary coherent communication, the signal set is given by

$$s_i(t) = \sqrt{2}\, A_c\, a_i(t)\ \cos(2\pi f_c t + \theta_i(t) + \phi)\, p_T(t)\ , \quad i = 0, 1$$

  where $a_i(t)$ is the *amplitude* modulation term, and $\theta_i(t)$ is the *phase* modulation term. The communications is said to be coherent since knowledge of carrier frequency f_c and phase ϕ is assumed at the receiver.

  ∘ If $a_0(t) \neq a_1(t)$ and $\theta_0(t) = \theta_1(t)$, then we have *amplitude shift keying* (ASK).
  ∘ If $\theta_0(t) \neq \theta_1(t)$ and $a_0(t) = a_1(t)$, then we have *phase shift keying* (PSK)

- **Binary Phase Shift Keying (BPSK):** This is the simplest form of passband digital communications. Here $a_0(t) = a_1(t) = p_T(t)$ and

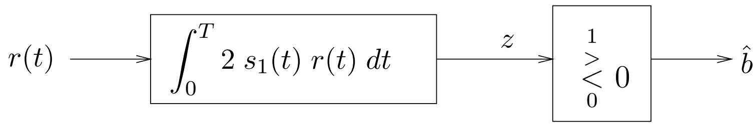
$$\theta_1(t) = 0,\ \theta_0(t) = \pi \ \text{ for } 0 \leq t \leq T\ .$$

  That is, the signal set is given by

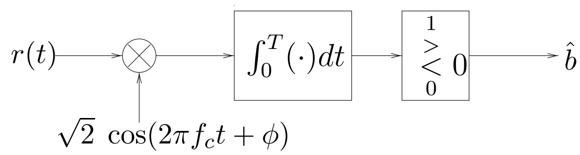
$$\begin{aligned} s_1(t) &= \sqrt{2}\, A_c\, p_T(t)\ \cos(2\pi f_c t + \phi) \\ s_0(t) &= \sqrt{2}\, A_c\, p_T(t)\ \cos(2\pi f_c t + \pi + \phi) = -\sqrt{2}\, A_c\, p_T(t)\ \cos(2\pi f_c t + \phi) = -s_1(t) \end{aligned}$$

  This is an *antipodal* signal set. Note that BPSK can also be viewed as BASK with $a_1(t) = -a_0(t) = p_T(t)$ and $\theta_1(t) = \theta_0(t) = 0$.

- **Optimum Receiver for BPSK:** Since BPSK is antipodal signaling, the optimum receiver correlates $r(t)$ with $2s_1(t)$ and compares the output to 0 for decision making as shown below:



Now since $\int_0^T 2\ s_1(t)\ r(t)\ dt = 2\sqrt{2}\ A_c \int_0^T \cos(2\pi f_c t + \phi)\ r(t)\ dt$ and scale factors do not affect the performance of the receiver, we may redraw the optimum receiver as shown below:



- **Performance of the Optimum Receiver for BPSK:**

$$\mathcal{E}_1 = \mathcal{E}_0 = \mathcal{E} = \int_0^T 2A_c^2 \cos^2(2\pi f_c t + \phi)\ dt = \cdots = A_c^2 T + A_c^2 \left[\frac{\sin(4\pi f_c t + 2\phi) - \sin(2\phi)}{4\pi f_c}\right]$$

If f_c is chosen such that $f_c T = n$ for some integer n or if we *ignore the "double frequency" terms*, the second term on the RHS equals 0, and we get $\mathcal{E} = A_c^2 T$. Thus

$$\mathrm{P_{e,opt}} = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}}\right) = Q\left(\sqrt{\frac{2A_c^2 T}{N_0}}\right)$$