ECE 359 **Spring 2003**

Handout # 9 March 11, 2003

NOTES ON RANDOM PROCESSES – I

The Gaussian Probability Density Function

This is the most important pdf for this course. It also called a *normal* pdf.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right].$$

It can be shown this f_X integrates to 1 (i.e., it is a valid pdf), and that the mean of the random variable X with the above pdf is m and the variance is σ^2.

The statement "X is Gaussian with mean m and variance σ^2" is compactly written as "$X \sim \mathcal{N}(m, \sigma^2)$."

The cdf corresponding to the Gaussian pdf is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(u-m)^2}{2\sigma^2}\right] du.$$

This integral cannot be computed in closed-form, but if we make the change of variabe $\frac{u-m}{\sigma} = v$ we get

$$F_X(x) = \int_{-\infty}^{\frac{x-m}{\sigma}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{v^2}{2}\right] dv = \Phi\left(\frac{x-m}{\sigma}\right),$$

where Φ is the cdf of a $\mathcal{N}(0,1)$ random variable, i.e.,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du.$$

Note that due to the symmetry of the Gaussian pdf,

$$\Phi(-x) = 1 - \Phi(x).$$

A closely related function to Φ is the Q function which is defined by:

$$Q(x) = 1 - \Phi(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2}\right] du.$$

Some end point properties of Φ and Q are given below:

$$Q(\infty) = \Phi(-\infty) = 0, \quad Q(-\infty) = \Phi(\infty) = 1, \quad Q(0) = \Phi(0) = 0.5$$

For computing the Q function in Matlab, we may use the Matlab functions `erf` or `erfc` after modifying them appropriately.

Jointly Gaussian Random Variables

Two random variables X and Y are said to be jointly Gaussian if their joint density satisfies the equation

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-m_X)^2}{\sigma_X^2} - \frac{2\rho(x-m_X)(y-m_Y)}{\sigma_X\sigma_Y} + \frac{(y-m_Y)^2}{\sigma_Y^2}\right]\right\}.$$

Note that the following properties hold:

- X is Gaussian with mean m_X and variance σ_X^2
- Y is Gaussian with mean m_Y and variance σ_Y^2
- The conditional densities $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ are also Gaussian
- ρ is the correlation coefficient between X and Y. If $\rho = 0$, then X and Y are independent.
- $Z = aX + bY$ is also Gaussian (what are the mean and variance of Z?)

The definition of jointly Gaussian random variables extends quite naturally to n variables $X_1, X_2, \ldots, X_n$. Let the vectors $\mathbf{X}$ and $\mathbf{m}$, and matrix $\mathbf{C}$ be defined by

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \quad \mathbf{m} = \mathrm{E}[\mathbf{X}] = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix} \quad \mathbf{C} = \mathrm{E}[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^\top] = \begin{bmatrix} C_{11} & C_{12} & \ldots & C_{1n} \\ C_{21} & C_{22} & \ldots & C_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n1} & C_{n2} & \ldots & C_{nn} \end{bmatrix}$$

where $m_i = \mathrm{E}[X_i]$ and $C_{ij} = \mathrm{cov}(X_i, X_j)$. Then the random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if their joint density is given by

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^\top \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right].$$

The statement "$X_1, X_2, \ldots, X_n$ are jointly Gaussian with mean $\mathbf{m}$ and covariance matrix $\mathbf{C}$" can be compactly written as "$\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$".

Properties of jointly Gaussian random variables include:

- Any subset of jointly Gaussian random variables is also jointly Gaussian.
- Any subset of jointly Gaussian random variables conditioned on any other subset of the original random variables is also jointly Gaussian.
- Jointly Gaussian random variables that are uncorrelated are also independent.
- Linear combinations of jointly Gaussian random variables are also jointly Gaussian. In particular, suppose we produce the vector $\mathbf{Y} = [Y_1 Y_2 \ldots Y_m]^\top$ using the linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{X}$, where $\mathbf{A}$ is an $m \times n$ matrix. Then,

  $$\mathbf{Y} \sim N\left(\mathbf{A}\mathbf{m}, \mathbf{A}\mathbf{C}\mathbf{A}^\top\right)$$

  i.e., $\mathbf{Y}$ is jointly Gaussian with mean $\mathbf{A}\mathbf{m}$, and covariance matrix $\mathbf{A}\mathbf{C}\mathbf{A}^\top$.

RANDOM PROCESSES

A random process is $X(t)$ simply a "random" signal or a random function of t. Just as in the definition of a random variable, there is an underlying sample space Ω in the definition of a random process, which means we should write $X(t)$ as $X(t;\omega)$, with $\omega \in \Omega$.

- For a fixed value ω_0 in the sample space Ω, $X(t;\omega_0)$ is simply a deterministic function of time. The function $X(t;\omega) \equiv x(t;\omega)$ is called a *sample function*. A generic sample function of the process $X(t)$ is usually denoted by the lower-case function $x(t)$.
- For for fixed t_0, $X(t_0;\omega)$ (or simply $X(t_0)$) is a random variable on Ω.

Example 1: Sinusoid with a random phase (explicit sample space)
Suppose $\Omega = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\}$ with probability distribution $\mathrm{P}(0) = \mathrm{P}(\frac{\pi}{2}) = \mathrm{P}(\pi) = \mathrm{P}(\frac{3\pi}{2}) = \frac{1}{4}$. Define a random process on Ω by

$$X(t,\omega) = a\cos(2\pi f_0 t + \omega)\ .$$

This random process has four sample functions which are sinusoids with different phases. Fixing t at 0 (say), we have that $X(0)$ is a discrete random variable with pmf given by:

$$\mathrm{P}\{X(0) = a\} = \mathrm{P}\{X(0) = -a\} = \frac{1}{4}\ , \text{ and } \mathrm{P}\{X(0) = 0\} = \frac{1}{2}\ .$$

Example 2: Sinusoid with a random phase (implicit sample space)
Suppose Θ is a random variable that is uniformly distributed on $[0, 2\pi]$. That is

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{if } 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Define

$$X(t) = a\cos(2\pi f_0 t + \Theta)$$

For each $\theta \in [0, 2\pi]$, $X(t,\theta)$ is a sinusoid with phase θ. For fixed $t = t_0$, $X(t_0)$ is a random variable; the cdf (pdf) of $X(t_0)$ can be computed (try it!).

A random process is completely characterized only if we know the joint cdf (or pdf) of $X(t_1), X(t_2), \ldots, X(t_n)$ for any integer n and any $t_1, t_2, \ldots, t_n$. But most of the time we will be interested in only first and second order properties. These are:

Mean Function: $\mathrm{E}[X(t)] = m_X(t)$.

Autocorrelation Function: $\mathrm{E}[X(t_1)X(t_2)] = R_X(t_1, t_2)$ or $\mathrm{E}[X(t+\tau)X(t)] = R_X(t+\tau, t)$.

Example 2 (continued) For the sinusoid with random phase,

$$m_X(t) = \int_{\theta=0}^{2\pi} \frac{1}{2\pi} a\cos(2\pi f_0 t + \theta) d\theta$$

and

$$R_X(t+\tau, t) = \int_{\theta=0}^{2\pi} \frac{1}{2\pi} a^2 \cos(2\pi f_0 + \theta)\cos(2\pi f_0(t+\tau) + \theta) d\theta$$

As we saw in class:

$$m_X(t) = 0 \qquad \text{and} \qquad R_X(t+\tau, t) = \frac{a^2}{2}\cos(2\pi f_0 \tau)$$