**ECE 459** **Fall 2000**

*Handout # 2* August 29, 2000

# HOMEWORK ASSIGNMENT 1

**Reading:** ECE 434 notes, lecture notes (lectures 1-3), Proakis (Chapter 4), papers referenced in lecture notes.

**Note:** The first 4 questions are meant to review your background in probability and random processes (ECE 434 material). The last 4 questions are on complex baseband representations.

**Due Date: Tuesday, September 5, 2000** (in class)

1. (17 pts total) *Random vectors and covariance matrices.*

   (a) (4 pts) Let $\mathbf{X}$ be a random $n$-vector with mean $\mathbf{m}$ and covariance matrix $\boldsymbol{\Sigma}$. Give expressions for the *mean* and *variance* of $\sum_{i=1}^{n} a_i Y_i$ in terms of $\mathbf{m}$ and $\boldsymbol{\Sigma}$.

   (b) (4 pts) Using the above result (or by other means), show that for mutually uncorrelated random variables, the variance of the sum is the sum of the variances.

   (c) (2 pts) Is the vector $[1 \ -1 \ -1]^\top$ a *right eigenvector* of the following matrix?

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

   (d) (7 pts) Consider a random vector $\mathbf{X} = [X_1 \ X_2]^\top$ with mean $[1 \ 2]^\top$ and covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

   Form the vector $\mathbf{Y} = [Y_1 \ Y_2]^\top$ as $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$. Find $\mathbf{A}$ and $\mathbf{b}$ such that $\mathbf{Y}$ is zero mean with covariance matrix $\mathbf{I}$. (*Hint:* Diagonalize $\boldsymbol{\Sigma}$.)

2. (15 pts total) *Bounds on the Q function.*

$$Q(x) = \int_x^\infty \frac{e^{-t^2/2}}{\sqrt{2\pi}} dt$$

   (a) (8 pts) For $x > 0$ show that the following upper and lower bounds hold for the $Q$ function:

$$\left(1 - \frac{1}{x^2}\right) \frac{e^{-x^2/2}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-x^2/2}}{x\sqrt{2\pi}}$$

   **Hint:** For the upper bound, write the integrand as a product of $1/t$ and $te^{-t^2/2}$, use integration by parts, and bound. For the lower bound, integrate by parts once more and bound.

(b) (7 pts) As you know from your undergraduate communications course, the bit error probability for BPSK signaling in additive white Gaussian noise (AWGN) with PSD $N_0/2$ is given by:

$$\mathrm{P_e} = Q\left(\sqrt{\frac{2\mathcal{E}_\mathrm{b}}{N_0}}\right)$$

where $\mathcal{E}_\mathrm{b}$ is the bit energy.

Plot the error probabililty $\mathrm{P_e}$ (on a log scale) versus signal-to-noise ratio $\mathcal{E}_\mathrm{b}/N_0$ (in dB) using Matlab or Mathematica. (You may need to use an appropriately modified version of the error function in these packages.) Consider $\mathcal{E}_\mathrm{b}/N_0$ ranging from $-5$ dB to 15 dB. Also plot the bounds and compare.

3. (20 pts total) *Rayleigh and Ricean Random Variables.* Let $X$ and $Y$ be independent $\mathcal{N}(0, \sigma^2)$ random variables, and define $R$ and $\Theta$ by:

$$R = \sqrt{X^2 + Y^2}\,, \text{ and } \Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

(Assume $\Theta \in [-\pi, \pi]$.)

(a) (4 pts) Find the joint pdf of $R$ and $\Theta$. Are $R$ and $\Theta$ independent?

(b) (4 pts) Find the marginal pdf's of $R$ and $\Theta$. (The pdf of $R$ is called a *Rayleigh* pdf).

(c) (4 pts) Find the pdf of $R^2$. (You should see that it is an *exponential*.)

(d) (8 pts) Now assume that $X \sim \mathcal{N}(a, \sigma^2)$ and $X \sim \mathcal{N}(b, \sigma^2)$ independent, where $a$ and $b$ are deterministic and possibly nonzero. Find the joint pdf of $R$ and $\Theta$ and from this the marginal pdf of $R$. Express the latter in terms of the modified Bessel function of the first kind:

$$I_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(x\cos\phi)d\phi$$

The pdf of $R$ in this case is called *Ricean.*

4. (18 pts total) *Random Processes*

(a) (4 pts) Let $\{X(t)\}$ be a WSS process with ACF $R_X(\tau)$. Find $\mathrm{E}\left[(X(1) + X(2))^2\right]$.

(b) (6 pts) Suppose $V$ is a zero-mean Gaussian random variable and define the processes $X(t) = Vt$ and $Y(t) = V^2 t$, for $-\infty < t < \infty$.
(a) Find the crosscorrelation function $R_{X,Y}(t + \tau, t)$.
(b) Are the two random processes jointly WSS?
(c) Are the two random processes uncorrelated?

(c) (4 pts) A random process $\{X(t)\}$ is given by

$$X(t) = \cos(2\pi f_0 t + \Theta)$$

where $\Theta$ is uniformly distributed on $[-\pi, \pi]$. Find the power spectral density $S_X(f)$.

(d) (4 pts) An ideal white noise process $\{n(t)\}$ with PSD $S_n(f) = \frac{N_0}{2}$ is input to linear system with impulse response $h(t) = e^{-t}u(t)$. Let $\{Y(t)\}$ denote the output process. Find $S_Y(f)$ and $R_Y(\tau)$.

5. (8 pts) Consider a bandpass signal $\{\tilde{s}(t)\}$ (centered around frequency $f_c$) that is passed through a bandpass LTI channel with impulse response $\{\tilde{h}(t)\}$, and let the output of the channel be $\{\tilde{y}(t)\}$. Let $\{\tilde{s}(t)\}$ and $\{\tilde{y}(t)\}$ be represented in complex baseband by $\{s(t)\}$ and $\{y(t)\}$, respectively.

   Show that $\{y(t)\}$ can be obtained by passing $\{s(t)\}$ through a complex baseband LTI channel with impulse response $\{h(t)\}$, and find $\{h(t)\}$ in terms of $\{\tilde{h}(t)\}$.

6. (10 pts) The complex random variable $Z = X + jY$ is zero mean and Gaussian but not necessarily proper. Show that

$$\mathrm{E}\exp(j\nu Z) = \exp(-\nu^2 \mathrm{E}Z^2/2)\ ,$$

   where $\nu$ can be assumed to be real (though this is not really necessary). This result is known as Vanderkulk's lemma and is similar to the characteristic function result for a real Gaussian random variable. Note that this gives the interesting result that $\mathrm{E}\exp(j\nu Z) = 1$ when $Z$ is proper complex Gaussian.

7. (12 pts) Let $\{n(t)\}$ be a zero-mean WSS bandpass process, and let $\{w(t) = w_I(t) + jw_Q(t)\}$ denote its complex baseband envelope. Show that the real random processes $\{w_{\mathrm{I}}(t)\}$ and $\{w_{\mathrm{Q}}(t)\}$ are jointly WSS.

8. **Extra Credit:** (5 pts) Prove the following theorem from lecture 3: If $\mathbf{Y} = \mathbf{Y}_I + j\mathbf{Y}_Q$ is a zero-mean proper complex Gaussian vector, then

$$p_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{\pi^n\,|\mathbf{\Sigma}_{\mathbf{Y}}|}\exp\left\{-\mathbf{y}^\dagger \mathbf{\Sigma}_{\mathbf{Y}}^{-1}\mathbf{y}\right\}.$$