ECE 459 **Fall 2000**

Handout # 4 September 27, 2000

HOMEWORK ASSIGNMENT 3

Reading: Lecture notes (lectures 8-11), Proakis (Chapter 5), papers/books referenced in lecture notes.

Due Date: Thursday, October 12, 2000 (in class)

1. (15 pts) *Estimation.*

   (a) (5 pts) Suppose the likelihood function for real-valued parameter λ is given by

   $$p_\lambda(y) = e^{-(y-\lambda)} \mathbb{1}_{\{y \geq \lambda\}} \,.$$

   Find $\hat{\lambda}_{\rm ML}(y)$. Now find $\hat{\lambda}_{\rm MAP}(y)$ under the assumption that $p_\Lambda(\lambda) = e^{-\lambda}\mathbb{1}_{\{\lambda \geq 0\}}$. Also find $\hat{\lambda}_{\rm MAP}(y)$ under the assumption that Λ is uniformly distributed on $[0, 1]$.

   (b) (10 pts) Suppose $\mathbf{Y} = [Y_1 \; Y_2 \; \cdots \; Y_n]$ is a vector of i.i.d. random variables with marginal pdf given $p_\lambda(\cdot)$ given in part (a). Find $\hat{\lambda}_{\rm ML}(\mathbf{y})$. Also find $\hat{\lambda}_{\rm MAP}(\mathbf{y})$ under the assumption that $p_\Lambda(\lambda) = e^{-\lambda}\mathbb{1}_{\{\lambda \geq 0\}}$.

2. (15 pts) *Detection in Discrete-Time Colored Gaussian Noise.* Consider the binary detection problem (i.e., $\lambda \in \{0, 1\}$) with likelihood function

   $$p_\lambda(\mathbf{y}) = \frac{1}{\pi^n \det(\mathbf{\Sigma})} \exp\left\{ -(\mathbf{y} - \boldsymbol{\mu}_\lambda)^\dagger \mathbf{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}_\lambda) \right\} \,, \quad \lambda = 0, 1 \,.$$

   (a) (5 pts) Show that for equal priors

   $$\hat{\lambda}_{\rm MPE} = \hat{\lambda}_{\rm ML} = \begin{cases} 1 & \text{if } B(\mathbf{y}) \geq d^2 \\ 0 & \text{otherwise} \end{cases}$$

   where

   $$B(\mathbf{y}) = 2{\rm Re}[(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^\dagger \mathbf{\Sigma}^{-1} (\mathbf{y} - \boldsymbol{\mu}_0)] \,, \text{ and } \; d^2 = (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^\dagger \mathbf{\Sigma}^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)$$

   (b) (x pts) Show that P_e for the MPE detector equals $Q(d/\sqrt{2})$.

3. (10 pts) *Noncoherent Demodulation of Linearly Modulated Signals.* The received signal for one symbol period for linear memoryless modulation on an ideal AWGN channel is given by:

   $$y(t) = \sqrt{\mathcal{E}_m} e^{j\theta_m} g(t) e^{j\phi} + w(t)$$

   where the phase offset ϕ is due to the delay introduced by channel. If ϕ is known at the receiver, we can correct for it (by say projecting $y(t)$ on $g(t)e^{j\phi}$ to produce the sufficient statistic) and suffer no loss in detection performance. However, if ϕ is not known, we may project $y(t)$ on $g(t)$ to get the sufficient statistic

   $$y = \sqrt{\mathcal{E}_m} e^{j\theta_m} e^{j\phi} + w$$

   where $w \sim \mathcal{CN}(0, N_0)$. Since ϕ is not of direct interest to the receiver, we treat it as a nuisance parameter. As we saw in class, there are two ways to deal with such parameters.

(a) (5 pts) Assume that $\phi \in [0, 2\pi]$, and find $\hat{m}_{\mathrm{ML}}(y)$ using the joint ML approach. Interpret your answer.

(b) (5 pts) Now assume that ϕ is a random variable that is uniformly distributed on $[0, 2\pi]$. Use the Bayesian approach to get an expression for $\hat{m}_{\mathrm{MAP}}(y)$ under equal priors on m. Simplify your answer as much as possible.

Note: You should see from this problem that noncoherent demodulation of linearly modulated signals is not a very good idea.

4. (15 pts) *Performance of MPSK.*

(a) (5 pts) Using the Intelligent Union Bound, show that the symbol error probability for MPSK signaling in AWGN is bounded by

$$\mathrm{P_e} \le 2Q\left(\sqrt{\frac{2\mathcal{E}_\mathrm{s}}{N_0}}\sin\frac{\pi}{M}\right).$$

(b) (10 pts) Now, derive the following exact expression for $\mathrm{P_e}$.

$$\mathrm{P_e} = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left[-\frac{\mathcal{E}_\mathrm{s}}{N_0}\frac{\sin^2(\pi/M)}{\sin^2\theta}\right] d\theta\,.$$

Hint: Shift the origin to the signal point under consideration and use polar co-ordinates with the appropriate limits of integration.

5. (30 pts) *Competing QAM Constellations.* Consider the three 8-ary QAM constellations shown in Figure 1.

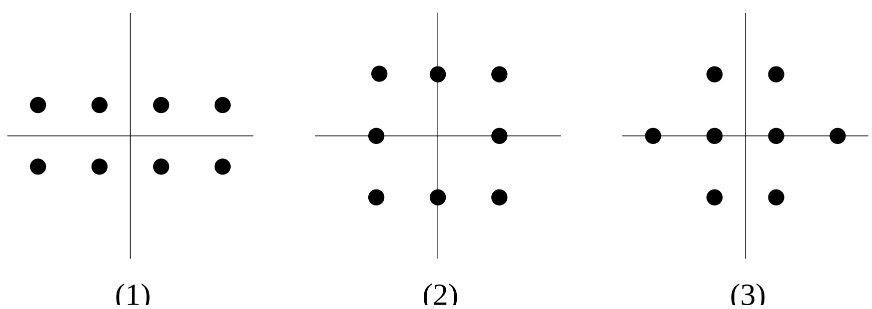

Figure 1: Signal constellations for Problem 5.

(a) (5 pts) Let the d_i denote the minimum distance for constellation i, $i = 1, 2, 3$. Find d_2 and d_3 in terms of d_1 so that all three constellations have the same average symbol energy $\mathcal{E}_\mathrm{s}$.

(b) (2 pts) For a given $\gamma_\mathrm{s} = \mathcal{E}_\mathrm{s}/N_0$, which constellation do you expect has the smallest $\mathrm{P_e}$ for a high SNR AWGN channel?

(c) (6 pts) Compute the nearest neighbor approximations for $\mathrm{P_e}$ for the three constellations.

(d) (7 pts) For each constellation, determine whether you can label the signal points using three bits so that nearest neighbors differ by at most one bit (Gray coding). If so, find such a labeling. If not, state why not and find a labeling that minimizes the bit transitions between neighbors.

(e) (10 pts) For the labelings found in part (d), compute the nearest neighbor approximation for the average bit error probability $\mathrm{P_b}$ as a function of the bit SNR $\gamma_\mathrm{b} = \mathcal{E}_\mathrm{b}/N_0$. Evaluate these approximations for $\gamma_\mathrm{b} = 10$ dB.

6. (15 pts) *Non-coherent Orthogonal Modulation.* Show that the probability of correctly demodulating the transmitted symbol for an M-ary non-coherent orthogonal modulation scheme is given by:

$$\mathrm{P_c} = \int_0^\infty x I_0\left(x\sqrt{\frac{2\mathcal{E}_\mathrm{s}}{N_0}}\right) \exp\left[-\left(\frac{x^2}{2} + \frac{\mathcal{E}_\mathrm{s}}{N_0}\right)\right] \left[1 - \exp\left(-\frac{x^2}{2}\right)\right]^{M-1} dx$$

Now use the binomial expansion on last term in the integrand, and the fact that a Ricean pdf integrates to 1, to show that the symbol error probability is given by

$$\mathrm{P_e} = \sum_{n=1}^{M-1} (-1)^{n+1} \binom{M-1}{n} \frac{1}{n+1} \exp\left[-\frac{n\mathcal{E}_\mathrm{s}}{(n+1)N_0}\right].$$