HOMEWORK ASSIGNMENT 4

Reading: Lecture notes (lectures 12-15), papers/books referenced in lecture notes.

Due Date: Thursday, November 2, 2000 (in class)

1. *ACF of Flat Fading Process* $\{E(t)\}$. Consider the flat fading process $\{E(t)\}$ for purely diffuse (Rayleigh) fading. We showed in class that for fixed t, $E(t)$ is a zero mean, unit variance PCG random variable with a Rayleigh envelope. Since $\{E(t)\}$ is Gaussian, all that is needed to characterize it completely (statistically) is its ACF.

   (a) Using equation (13.7) of the lecture notes, show that

   $$R_E(t+\tau, t) \approx \sum_n \beta_n^2 e^{j2\pi f_m \tau \cos\theta_n} .$$

   This means that $\{E(t)\}$ is (approximately) a stationary process.

   (b) Based on your answer to part (a), find the ACF's and crosscorrelation functions of $\{E_I(t)\}$ and $\{E_Q(t)\}$. Are $\{E_I(t)\}$ and $\{E_Q(t)\}$ independent processes?

   We can consider β_n^2 to be the fraction of power gain of the channel corresponding to path n (or angle of arrival θ_n). Our goal now is to characterize the power gain in the multipath environment as a function of angle of arrival. Since the $\{\beta_n\}$ have been normalized so that $\sum_n \beta_n^2 = 1$, we can define an angular power gain *density* $p(\theta)$ as

   $$p(\theta) = \sum_n \beta_n^2 \, \delta(\theta - \theta_n) . \tag{1}$$

   Then we can write

   $$R_E(\tau) = \int_{-\pi}^{\pi} p(\theta) e^{j2\pi f_m \tau \cos\theta} d\theta . \tag{2}$$

   Note that $R_E(\tau)$ is strong function of angular power gain density.

   Consider the situation where the propagation environment is such that power is received roughly uniformly from all directions. We may model such scattering as continuous and *isotropic* and set $p(\theta) = 1/2\pi$ for all θ.

   (c) Show that $R_E(\tau)$ for isotropic scattering equals $J_0(2\pi f_m \tau)$, where $J_0(\cdot)$ is the *zeroth order Bessel function of the first kind.*

   $$J_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x \cos\theta) d\theta.$$

   Also find the the ACF's and crosscorrelation functions of $\{E_I(t)\}$ and $\{E_Q(t)\}$. Are $\{E_I(t)\}$ and $\{E_Q(t)\}$ independent processes in this case?

2. *Diversity.* Consider BPSK with channel gain a, i.e., the received signal is

$$r(t) = \pm a\sqrt{\mathcal{E}}g(t) + w(t), \quad 0 \le t \le T,$$

where $\{w(t)\}$ is a zero-mean complex WGN process with PSD N_0, $g(t)$ is a unit energy signal, and the channel gain a is random with probability mass function

$$\mathrm{P}\{a = 0\} = 0.1 \quad \text{and} \quad \mathrm{P}\{a = 2\} = 0.9.$$

(a) Determine the average probability of error $\overline{\mathrm{P}}_{\mathrm{e}}$ for MPE demodulation.

(b) What value does $\overline{\mathrm{P}}_{\mathrm{e}}$ approach as $\mathcal{E}/N_0$ approaches infinity?

(c) Suppose the same signal is transmitted on two statistically *independent* channels with gains a_1 and a_2, where

$$\mathrm{P}\{a_1 = 0\} = \mathrm{P}\{a_2 = 0\} = 0.1 \quad \text{and} \quad \mathrm{P}\{a_1 = 2\} = \mathrm{P}\{a_2 = 2\} = 0.9.$$

The additive noises on the two channels are also independent and identically distributed. The demodulator employs a matched filter for each channel and adds the two filter outputs to form the decision variable (which is compared to 0 for decision-making). Determine $\overline{\mathrm{P}}_{\mathrm{e}}$ in this case.

(d) For the case in part (c), what value does $\overline{\mathrm{P}}_{\mathrm{e}}$ approach when $\mathcal{E}/N_0$ approaches infinity?

3. *Useful results.* Show that

(a)

$$\int_0^\infty Q(\sqrt{x})\frac{e^{-x/\gamma}}{\gamma}dx = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{2+\gamma}}\right).$$

(b) For $t > 0$,

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta.$$

This is an alternative, and useful form, for the $Q(\cdot)$ function.

4. *MPSK in Rayleigh fading.*

(a) Based on the expression for P_{e} for MPSK (without fading) derived in problem 4(b) of HW#3, show that the average symbol error probability $\overline{\mathrm{P}}_{\mathrm{e}}$ for Rayleigh fading is given in closed form by

$$\overline{\mathrm{P}}_{\mathrm{e}} = \left(1 - \frac{1}{M}\right) - \frac{1}{\sqrt{1+a^2}}\left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\cot\pi/M}{\sqrt{1+a^2}}\right)\right],$$

where $\overline{\gamma}_{\mathrm{b}}$ is the average bit SNR and $a^2 = \frac{1}{\overline{\gamma}_{\mathrm{b}}k\sin^2\pi/M}$.

Hint: You may need to use the following integral

$$\int_{\theta_1}^{\theta_2} \frac{1}{\mathrm{cosec}^2\theta + a^2}d\theta = \frac{1}{a^2}\left[\frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{\cot\theta}{\sqrt{1+a^2}}\right) - \left(\frac{\pi}{2} - \theta\right)\right]_{\theta_1}^{\theta_2} \quad \text{for } 0 \le \theta_1 \le \theta_2 \le \pi/2.$$

(b) Assume Gray coding, so that $\overline{\mathrm{P}}_{\mathrm{b}} \approx \overline{\mathrm{P}}_{\mathrm{e}}/\nu$, and plot $\overline{\mathrm{P}}_{\mathrm{b}}$ as a function of $\overline{\gamma}_{\mathrm{b}}$ for $M = 2, 4, 8, 16$. Use a log scale for the $\overline{\mathrm{P}}_{\mathrm{b}}$ axis and show $\overline{\gamma}_{\mathrm{b}}$ values in dB between 0 and 40 dB. Compare with the AWGN plot given on the web site.

5. *Non-coherent M-ary orthogonal signaling in Ricean fading*:

   (a) Using the expression for $\mathrm{P_e}$ in Problem 6 of HW#3, show that $\overline{\mathrm{P}}_\mathrm{e}$ under Ricean fading with Rice factor κ is given by

   $$\overline{\mathrm{P}}_\mathrm{e} = \sum_{n=1}^{M-1} (-1)^{n+1} \binom{M-1}{n} \frac{\tilde{p}(n, \overline{\gamma}_\mathrm{b})}{n+1},$$

   where

   $$\tilde{p}(n, \overline{\gamma}_\mathrm{b}) = \frac{\kappa + 1}{\kappa + 1 + \frac{nk}{n+1}\overline{\gamma}_\mathrm{b}} \exp\left(- \frac{\frac{\kappa n k}{n+1}\overline{\gamma}_\mathrm{b}}{\kappa + 1 + \frac{nk}{n+1}\overline{\gamma}_\mathrm{b}} \right)$$

   (b) For each of $M = 2, 4, 8$, plot $\overline{\mathrm{P}}_\mathrm{b}$ versus $\overline{\gamma}_\mathrm{b}$ (ranging from 5 to 40 dB) for $\kappa = 0, 5, 10$. Compare with the corresponding AWGN curves given to you in class.

6. *Nakagami-m fading.* The first order statistics of a flat fading channel are sometimes approximated by a pdf from the Nakagami-m family:

   $$p_\alpha(x) = \frac{2m^m x^{2m-1}}{\Gamma(m) a^m} \exp\left(-\frac{mx^2}{a} \right) 1\!\!1_{\{x>0\}}, \quad m > 0.5$$

   where $a = \mathrm{E}[\alpha^2] = 1$, and $\Gamma(\cdot)$ is the Gamma function which is defined by the integral $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, for $x > 0$. (Properties of $\Gamma(\cdot)$ include: $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(0.5) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(n+1) = n!$, for positive integer n.)

   Note: If m is positive integer, which means that $\Gamma(m) = (m-1)!$, the above p.d.f. is a central chi-squared distribution with $2m$ degrees of freedom, which is also the p.d.f. of the sum of m independent and identically distributed exponential random variables – sometimes called an Erlang distribution.

   (a) Show that the pdf of γ_b is given by

   $$p_{\gamma_\mathrm{b}}(x) = \left(\frac{m}{\overline{\gamma}_\mathrm{b}} \right)^m \frac{x^{m-1}}{\Gamma(m)} \exp\left(-\frac{mx}{\overline{\gamma}_\mathrm{b}} \right) 1\!\!1_{\{x>0\}}.$$

   (b) If m is a positive integer, show that the c.d.f. of γ_b is given by:

   $$F_{\gamma_\mathrm{b}}(x) = \int_0^x p_{\gamma_\mathrm{b}}(y) dy = 1 - \sum_{i=0}^{m-1} \left(\frac{m}{\overline{\gamma}_\mathrm{b}} \right)^i \frac{x^i}{i!} \exp\left(-\frac{mx}{\overline{\gamma}_\mathrm{b}} \right).$$

   (c) Now show that if m is a positive integer, $\overline{\mathrm{P}}_\mathrm{b}$ for BPSK signaling in slow, flat Nakagami-m fading is given by

   $$\overline{\mathrm{P}}_\mathrm{b} = \frac{1}{2} - \frac{\tilde{\sigma}}{2} \sum_{i=0}^{m-1} \left(\frac{m\tilde{\sigma}^2}{4\overline{\gamma}_\mathrm{b}} \right)^i \frac{(2i)!}{(i!)^2},$$

   where $\tilde{\sigma} = \sqrt{\overline{\gamma}_\mathrm{b}/(\overline{\gamma}_\mathrm{b} + m)}$.
   Hint: You may want to use the fact that the even moments of a $\mathcal{N}(0,1)$ random variable X are given by:

   $$\mathrm{E}[X^{2i}] = \frac{(2i)!}{i!\, 2^i}.$$