HOMEWORK ASSIGNMENT 1

1. *Vanderkulk's Lemma.* The complex random variable $Z = X + jY$ is zero mean and Gaussian but not necessarily proper. Show that

$$\mathrm{E}\exp(j\nu Z) = \exp(-\nu^2 \mathrm{E}Z^2/2) ,$$

where ν can be assumed to be real (though this is not really necessary). This result is known as Vanderkulk's lemma and is similar to the characteristic function result for a real Gaussian random variable. Note that this gives the interesting result that E $\exp(j\nu Z) = 1$ when Z is proper complex Gaussian.

2. (Optional) *Properness of a PCG Vector.* Prove Result B.2 in the class notes: Let $\boldsymbol{Y} = \boldsymbol{Y}_I + j\boldsymbol{Y}_Q$ be proper complex and Gaussian, i.e. $\boldsymbol{Y}_I, \boldsymbol{Y}_Q$ are jointly Gaussian. Then

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y}) &:= p_{\boldsymbol{Y}_I \boldsymbol{Y}_Q}(\boldsymbol{y}_I, \boldsymbol{y}_Q) \\ &= \frac{1}{\pi^n |\boldsymbol{\Sigma}_{\boldsymbol{Y}}|} \exp\left\{ -(\boldsymbol{y} - \boldsymbol{m}_Y)^\dagger \boldsymbol{\Sigma}_Y^{-1} (\boldsymbol{y} - \boldsymbol{m}_Y) \right\} \end{aligned}$$

Note: This problem is optional, and is meant for brave souls who wish to explore the dark world of Matrix Algebra ◡̈ The solution can be found in the paper by Neeser and Massey.

3. *Cellular Area Reliability.* Read Section 2.5.1 of the notes on area reliability and derive equation (2.29), i.e., show that

$$F_{\text{area}} = Q(a + b\ln R) + \frac{\exp\left(\frac{2}{b^2} - \frac{2a}{b}\right)}{R^2} \left[1 - Q\left(a + b\ln R - \frac{2}{b} \right) \right] .$$

Also derive equation (2.31), i.e., show that

$$F_{\text{area}} = F_{\text{edge}} + e^{\frac{2}{b^2}} e^{-\frac{2}{b} Q^{-1}(F_{\text{edge}})} \left(1 - Q\left(Q^{-1}(F_{\text{edge}}) - \frac{2}{b} \right) \right)$$

4. *Signal strength prediction.* Consider a mobile that is moving on a straight line path (not necessarily radial) at a constant velocity v. In order to make handoff decisions, the mobile periodically takes pilot power measurements from neighboring BS's. Let us assume that these power measurements are averaged to remove multipath fluctuations, so the the resulting sampled measurements only have a median component and shadow fading. The k-th sample value of the pilot power (in dBm) from a particular BS is given by:

$$P_{r,k}[\text{dBm}] = \bar{P}_r(d_k) + Z_k = A_t - B\log d_k + Z_k,$$

where d_k is the distance from the BS at the k-th sampling time, and A_t includes the transmitted pilot power. Note that the d_k values are not necessarily equally spaced.

Let us assume isotropic shadow fading with exponential ACF. Since the velocity vector is constant, the random process $\{Z_k, k = 1, 2, \ldots\}$ is a stationary first-order *auto-regressive* (AR) process with

$$\mathrm{E}[Z_k Z_{k+m}] = \sigma_Z^2 a^{|m|},$$

where $a = \exp(-vt_s/D_c)$, t_s is the sampling time, and D_c is the correlation distance.

Handoff decisions are often based on signal strength prediction. Our goal here is to find the MMSE predictor of $P_{r,k+1}$ based on $P_{r,1}, P_{r,2}, \ldots, P_{r,k}$.

(a) Under the assumption that the d_k values are known, show that

$$\hat{P}_{r,k+1}^{\text{MMSE}} = \text{E}[P_{r,k+1}|P_{r,1}, P_{r,2}, \ldots, P_{r,k}] = aP_{r,k} + (1-a)A_t - B\log\left(\frac{d_{k+1}}{d_k^a}\right)$$

and that the corresponding mean-squared error

$$\text{MSE} = \text{Var}[P_{r,k+1}|P_{r,1}, P_{r,2}, \ldots, P_{r,k}] = (1-a^2)\sigma_Z^2.$$

Hint: You don't need to solve the Yule-Walker equations to find the MMSE solution in this special case. Use the AR-1 property of the $\{Z_k\}$ to find the solution directly.

(b) Discuss how you might address the prediction problem if the d_k values were unknown.

5. *Moments of lognormals.* Suppose X is a lognormal random variable with mean m_X and second moment δ_X, and suppose $Y = 10\log X$ has mean m_X and variance σ_Y^2.

(a) Show that

$$m_X = \exp\left(\frac{(\beta\sigma_Y)^2}{2}\right)\exp(\beta m_Y) \quad \text{and}$$
$$\delta_X = \exp\left(2(\beta\sigma_Y)^2\right)\exp(2\beta m_Y),$$

where $\beta = \ln(10)/10$.

(b) Also, show that

$$m_Y = 20\log m_X - 5\log\delta_X, \quad \text{and that}$$

$$\sigma_Y^2 = \frac{1}{\beta}(10\log\delta_X - 20\log m_X).$$

6. *Outage with macrodiversity.* Consider a mobile at the midpoint between two base stations in a cellular network. The received signals (in dB-W) from the base stations are given by

$$\begin{aligned} P_{r,1} &= A_t - B\log(D/2) + Z_1, \\ P_{r,2} &= A_t - B\log(D/2) + Z_2, \end{aligned}$$

where Z_1 and Z_2 are $\mathcal{N}(0, \sigma^2)$ random variables.

We define outage with macrodiversity to be the event that both $P_{r,1}$ and $P_{r,2}$ fall below a pre-specified threshold P_{thresh}.

(a) If Z_1 and Z_2 are independent, show that the outage probability is given by

$$\text{P}_{\text{out}} = \left[Q\left(\frac{\Delta}{\sigma}\right)\right]^2,$$

where

$$\Delta \triangleq A_t - B\log(D/2) - P_{\text{thresh}}$$

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(b) Now suppose Z_1 and Z_2 are correlated in the following way.

$$Z_1 = aY_1 + bY \quad \text{and} \quad Z_2 = aY_2 + bY,$$

where Y, Y_1 and Y_2 are *independent* $\mathcal{N}(0, \sigma^2)$ random variables, and a and b are such that $a^2 + b^2 = 1$.

Show that

$$\mathrm{P}_{\mathrm{out}} = \int_{-\infty}^{\infty} \left[Q\left(\frac{\Delta + by\sigma}{a\sigma} \right) \right]^2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

(c) Compare the outage probabilities of (i) and (ii) for the special case of $a = b = 1/\sqrt{2}$, $\sigma = 8$ and $\Delta = 5$. (Use a numerical integration routine for $\mathrm{P}_{\mathrm{out}}$ of (ii).)

7. *Squared-envelope correlation for isotropic fading:*

(a) Prove that the squared-envelope covariance function for a *Rayleigh* fading process $\{E(t)\}$ is given by:

$$C_{\alpha^2}(\tau) = |R_E(\tau)|^2$$

Hint: It may be easier to first show that $R_{\alpha^2}(\tau) = |R_E(\tau)|^2 + 1$. You may find the following result to be useful: if X_1, X_2, X_3, X_4 are jointly Gaussian *zero-mean* random variables, then

$$\mathrm{E}[X_1 X_2 X_3 X_4] = \mathrm{E}[X_1 X_2]\mathrm{E}[X_3 X_4] + \mathrm{E}[X_1 X_3]\mathrm{E}[X_2 X_4] + \mathrm{E}[X_1 X_4]\mathrm{E}[X_2 X_3].$$

(b) Prove that the squared-envelop covariance function for a *Ricean* fading process $\{E(t)\}$ is given by:

$$C_{\alpha^2}(\tau) = \left(\frac{1}{\kappa + 1} \right)^2 \left[|R_{\check{E}}(\tau)|^2 + 2\kappa \mathrm{Re}\left[R_{\check{E}}(\tau) e^{-j2\pi f_{\max}\tau\cos\theta_0} \right] \right],$$

where κ is the Rice factor and θ_0 is the angle of arrival of the specular component.

8. *Power Spectrum of Rayleing Fading Process.* Consider a Rayleigh fading environment with angular power density $p(\theta)$.

(a) Show that:

$$S_E(f) = \int_0^{\pi} [p(\theta) + p(-\theta)]\delta(f - f_{\max}\cos\theta)d\theta$$

(b) Now show that $S_E(f)$ has the closed-form expression:

$$S_E(f) = \begin{cases} \dfrac{p(\cos^{-1}(f/f_{\max})) + p(-\cos^{-1}(f/f_{\max}))}{\sqrt{f_{\max}^2 - f^2}} & \text{for } |f| < f_{\max} \\ 0 & \text{otherwise} \end{cases}$$

(c) Specialize this result for isotropic Rayleigh fading, i.e., $p(\theta) = 1/2\pi$.