ECE 559 **Fall 2007**

Handout # 4 September 20, 2007

SOLUTIONS TO HOMEWORK ASSIGNMENT 1

1. *Vanderkulk's Lemma.* The complex random variable $Z = X + jY$ is zero mean and Gaussian but not necessarily proper. Show that

$$\mathrm{E}\,\exp(j\nu Z) = \exp(-\nu^2 \mathrm{E}Z^2/2)\ ,$$

where ν can be assumed to be real (though this is not really necessary). This result is known as Vanderkulk's lemma and is similar to the characteristic function result for a real Gaussian random variable. Note that this gives the interesting result that E $\exp(j\nu Z) = 1$ when Z is proper complex Gaussian.

Solution. By definition

$$p_Z(z) \stackrel{\Delta}{=} p_{X,Y}(x,y) = p_{X|Y}(x|y)p_Y(y)$$

Since X and Y are zero mean and jointly Gaussian, X is conditionally Gaussian given Y, with conditional mean and variance:

$$m_1(y) = \frac{\sigma_X\ \rho\ y}{\sigma_Y}, \quad \sigma_1^2 = \sigma_X^2(1-\rho^2)$$

where $\rho = \mathsf{E}[XY]/\sigma_X\sigma_Y$.

Now, we have the following result on the characteristic function of Gaussian random variables. If V is a real-valued Gaussian random variable with mean m and variance σ^2, then

$$\mathsf{E}[e^{j\theta V}] = e^{j\theta m\ -\theta^2\sigma^2/2} \tag{1}$$

Note that (1) holds even when θ is complex.

Now,

$$\mathsf{E}[e^{j\nu Z}] = \mathsf{E}[e^{j\nu X - \nu Y}] = \mathsf{E}_Y\left[\mathsf{E}_{X|Y}[e^{j\nu X}]\ e^{-\nu Y}\right]$$

Applying (1) to the inner expectation, we get

$$\mathsf{E}_{X|Y}[e^{j\nu X}] = e^{j\nu m_1(Y) - \nu^2\sigma_1^2/2}$$

which implies that

$$\mathsf{E}[e^{j\nu Z}] = e^{\frac{\nu^2\sigma_1^2}{2}}\ \mathsf{E}_Y\left[e^{j\nu\left(\frac{\sigma_X\rho}{\sigma_Y}+j\right)Y}\right]$$

Now apply (1) to Y with $\theta = \nu\left(\frac{\sigma_X\rho}{\sigma_Y} + j\right)$ and we get

$$\mathsf{E}[e^{j\nu Z}] = e^{\frac{\nu^2\sigma_1^2}{2}}\ e^{-\frac{\nu^2}{2}\left(\frac{\sigma_X\rho}{\sigma_Y}+j\right)^2\sigma_Y^2} = e^{-\frac{\nu^2}{2}(\sigma_X^2+\sigma_Y^2-2\rho\sigma_X\sigma_Y j)} = e^{-\frac{\nu^2\mathsf{E}[Z^2]}{2}}$$

2. (Optional) *Properness of a PCG Vector.* Prove Result B.2 in the class notes: Let $\boldsymbol{Y} = \boldsymbol{Y}_I + j\boldsymbol{Y}_Q$ be proper complex and Gaussian, i.e. $\boldsymbol{Y}_I, \boldsymbol{Y}_Q$ are jointly Gaussian. Then

$$\begin{aligned} p_{\boldsymbol{Y}}(\boldsymbol{y}) &:= p_{\boldsymbol{Y}_I\boldsymbol{Y}_Q}(\boldsymbol{y}_I, \boldsymbol{y}_Q) \\ &= \frac{1}{\pi^n|\boldsymbol{\Sigma}_{\boldsymbol{Y}}|}\exp\left\{-(\boldsymbol{y}-\boldsymbol{m}_Y)^\dagger\boldsymbol{\Sigma}_Y^{-1}(\boldsymbol{y}-\boldsymbol{m}_Y)\right\} \end{aligned}$$

Note: This problem is optional, and is meant for brave souls who wish to explore the dark world of Matrix Algebra ◡̈ The solution can be found in the paper by Neeser and Massey.

3. *Cellular Area Reliability.* Read Section 2.5.1 of the notes on area reliability and derive equation (2.29), i.e., show that

$$F_{\text{area}} = Q(a + b \ln R) + \frac{\exp\left(\frac{2}{b^2} - \frac{2a}{b}\right)}{R^2} \left[1 - Q\left(a + b\ln R - \frac{2}{b}\right)\right] .$$

Also derive equation (2.31), i.e., show that

$$F_{\text{area}} = F_{\text{edge}} + e^{\frac{2}{b^2}} e^{-\frac{2}{b}Q^{-1}(F_{\text{edge}})} \left(1 - Q\left(Q^{-1}(F_{\text{edge}}) - \frac{2}{b}\right)\right)$$

Solution:

$$F_{\text{area}} = \frac{2}{R^2}\int_0^R Q(a + b\ln\rho)\rho d\rho = \frac{2}{bR^2}\int_{-\infty}^{a+b\ln R} Q(t)e^{\frac{2(t-a)}{b}} dt = \frac{2}{bR^2}e^{-\frac{2a}{b}}\int_{-\infty}^{a+b\ln R} Q(t)e^{\frac{2t}{b}} dt$$

Defining I to be the last integral in the previous equation, we get

$$\begin{aligned} I &= \int_{-\infty}^{a+b\ln R} Q(t)e^{\frac{2t}{b}} dt \\ &\stackrel{a}{=} Q(t)\frac{b}{2}e^{\frac{2t}{b}}\Big|_{-\infty}^{a+b\ln R} + \frac{b}{2\sqrt{2\pi}}\int_{-\infty}^{a+b\ln R} e^{-\frac{t^2}{2}}e^{\frac{2t}{b}} dt \\ &= \frac{b}{2}Q(a+b\ln R)e^{\frac{2(a+b\ln R)}{b}} + \frac{b}{2}e^{\frac{2}{b^2}}\int_{-\infty}^{a+b\ln R} \frac{e^{-\frac{(t-\frac{2}{b})^2}{2}}}{\sqrt{2\pi}} dt \\ &= \frac{b}{2}Q(a+b\ln R)R^2 e^{\frac{2a}{b}} + \frac{b}{2}e^{\frac{2}{b^2}}\left(1 - Q\left(a + b\ln R - \frac{2}{b}\right)\right) \end{aligned}$$

where in a we used integration by parts. Substituting this result back into the previous equation

$$F_{\text{area}} = \frac{2}{bR^2}e^{-\frac{2a}{b}} I = Q(a+b\ln R) + \frac{e^{\frac{2}{b^2}-\frac{2a}{b}}}{R^2}\left(1 - Q\left(a + b\ln R - \frac{2}{b}\right)\right).$$

From the definition of F_{edge} we have $a + b\ln R = Q^{-1}(F_{\text{edge}})$. Thus

$$\frac{e^{-\frac{2a}{b}}}{R^2} = \frac{e^{-\frac{2}{b}\left(Q^{-1}(F_{\text{edge}}) - b\ln R\right)}}{R^2} = e^{-\frac{2}{b}\left(Q^{-1}(F_{\text{edge}})\right)}\frac{e^{2\ln R}}{R^2}$$

and

$$F_{\text{area}} = F_{\text{edge}} + e^{\frac{2}{b^2}} e^{-\frac{2}{b}Q^{-1}(F_{\text{edge}})} \left(1 - Q(Q^{-1}(F_{\text{edge}}) - \frac{2}{b})\right)$$

Note that this is independent of R and a.

4. *Signal strength prediction.* Consider a mobile that is moving on a straight line path (not necessarily radial) at a constant velocity v. In order to make handoff decisions, the mobile periodically takes pilot power measurements from neighboring BS's. Let us assume that these power measurements are averaged to remove multipath fluctuations, so the resulting sampled measurements only have a median component and shadow fading. The k-th sample value of the pilot power (in dBm) from a particular BS is given by:

$$P_{r,k}[\text{dBm}] = \bar{P}_r(d_k) + Z_k = A_t - B\log d_k + Z_k,$$

where d_k is the distance from the BS at the k-th sampling time, and A_t includes the transmitted pilot power. Note that the d_k values are not necessarily equally spaced.

Let us assume isotropic shadow fading with exponential ACF. Since the velocity vector is constant, the random process $\{Z_k, k = 1, 2, \ldots\}$ is a stationary first-order *auto-regressive* (AR) process with

$$\mathrm{E}[Z_k Z_{k+m}] = \sigma_Z^2 a^{|m|},$$

where $a = \exp(-vt_s/D_c)$, t_s is the sampling time, and D_c is the correlation distance.

Handoff decisions are often based on signal strength prediction. Our goal here is to find the MMSE predictor of $P_{r,k+1}$ based on $P_{r,1}, P_{r,2}, \ldots, P_{r,k}$.

(a) Under the assumption that the d_k values are known, show that

$$\hat{P}_{r,k+1}^{\mathrm{MMSE}} = \mathrm{E}[P_{r,k+1}|P_{r,1}, P_{r,2}, \ldots, P_{r,k}] = aP_{r,k} + (1-a)A_t - B\log\left(\frac{d_{k+1}}{d_k^a}\right)$$

and that the corresponding mean-squared error

$$\mathrm{MSE} = \mathrm{Var}[P_{r,k+1}|P_{r,1}, P_{r,2}, \ldots, P_{r,k}] = (1-a^2)\sigma_Z^2.$$

Hint: You don't need to solve the Yule-Walker equations to find the MMSE solution in this special case. Use the AR-1 property of the $\{Z_k\}$ to find the solution directly.

Solution:

$$\begin{aligned} P_{r,k+1} &= A_t + B\log d_{k+1} + Z_{k+1} \\ &= A_t + B\log d_{k+1} + aZ_k + \sigma_Z\sqrt{1-a^2}W_k \\ &= a\left(A_t + B\log d_k + Z_k\right) + (1-a)A_t + B\log d_{k+1} - aB\log d_k + \sigma_Z\sqrt{1-a^2}W_k \end{aligned}$$

Thus,

$$P_{r,k+1} = aP_{r,k} + (1-a)A_t + B\log\frac{d_{k+1}}{d_k^a} + \sigma_Z\sqrt{1-a^2}W_k.$$

By the AR-1 model construction, W_k is independent of $(P_{r,k}, \ldots, P_{r,1})$. This implies that $P_{r,k+1}$ given $P_{r,k}$ is independent of $(P_{r,k-1}, \ldots, P_{r,1})$, and

$$\mathrm{E}\left[P_{r,k+1}|P_{r,k}, \ldots, P_{r,1}\right] = \mathrm{E}\left[P_{r,k+1}|P_{r,k}\right] = aP_{r,k} + (1-a)A_t + B\log\frac{d_{k+1}}{d_k^a}.$$

Similarly,

$$\mathrm{Var}\left[P_{r,k+1}|P_{r,k}, \ldots, P_{r,1}\right] = \mathrm{Var}\left[P_{r,k+1}|P_{r,k}\right] = \mathrm{E}\left[\left(\sigma_Z\sqrt{1-a^2}\right)^2 W_k^2\right] = \sigma_Z^2(1-a^2)$$

(b) Discuss how you might address the prediction problem if the d_k values were unknown.

Solution: The prediction problem is considerably harder when d_k is unknown. One way to approach the problem is to use a velocity estimate based on measurements of doppler frequency to first obtain an estimate of the distance $\Delta\rho$ between consecutive samples. We may then use a "curve fitting" technique to obtain an estimate of d_k based on all of the signal strength measurements. Thus, the best estimate of $P_{r,k+1}$ based on $(P_{r,k}, \ldots, P_{r,1})$ will be a function of all the measurements, not just the most recent one.

5. *Moments of lognormals.* Suppose X is a lognormal random variable with mean m_X and second moment δ_X, and suppose $Y = 10\log X$ has mean m_X and variance σ_Y^2.

   (a) Show that

$$m_X = \exp\left(\frac{(\beta\sigma_Y)^2}{2}\right)\exp(\beta m_Y) \quad \text{and}$$

$$\delta_X = \exp\left(2(\beta\sigma_Y)^2\right)\exp(2\beta m_Y),$$

   where $\beta = \ln(10)/10$.

   Solution: The characteristic function of Y is given by

$$\phi_Y(s) = \mathrm{E}[e^{sY}] = e^{sm_Y + \frac{s^2\sigma_Y^2}{2}}$$

   Thus,

$$m_X = \mathrm{E}[e^{\beta Y}] = \phi_Y(\beta) = e^{\beta m_Y + \frac{\beta^2\sigma_Y^2}{2}}$$

   and

$$\delta_X = \mathrm{E}[X^2] = \phi_Y(2\beta) = e^{2\beta m_Y + 2\beta^2\sigma_Y^2}.$$

   (b) Also, show that

$$m_Y = 20\log m_X - 5\log\delta_X, \quad \text{and that}$$

$$\sigma_Y^2 = \frac{1}{\beta}(10\log\delta_X - 20\log m_X).$$

   Solution: Taking the log of the previous equations, we get

$$\begin{aligned}
10\log m_X &= \frac{\ln m_X}{\beta} = \frac{\beta\sigma_Y^2}{2} + m_Y \\
10\log\delta_X &= \frac{\ln\delta_X}{\beta} = 2\beta\sigma_Y^2 + 2m_Y.
\end{aligned}$$

   Solving for m_Y and σ_Y^2, we obtain

$$\begin{aligned}
m_Y &= 20\log m_X - 5\log\delta_X \\
\sigma_Y^2 &= \frac{1}{\beta}\left(10\log\delta_X - 20\log m_X\right).
\end{aligned}$$

6. *Outage with macrodiversity.* Consider a mobile at the midpoint between two base stations in a cellular network. The received signals (in dB-W) from the base stations are given by

$$\begin{aligned}
P_{r,1} &= A_t - B\log(D/2) + Z_1, \\
P_{r,2} &= A_t - B\log(D/2) + Z_2,
\end{aligned}$$

   where Z_1 and Z_2 are $\mathcal{N}(0,\sigma^2)$ random variables.

   We define outage with macrodiversity to be the event that both $P_{r,1}$ and $P_{r,2}$ fall below a pre-specified threshold P_{thresh}.

(a) If Z_1 and Z_2 are independent, show that the outage probability is given by

$$\mathrm{P}_{\text{out}} = \left[Q\left(\frac{\Delta}{\sigma} \right) \right]^2,$$

where

$$\Delta \triangleq A_t - B\log(D/2) - P_{\text{thresh}}$$

is the fade margin at the edge of the cell.

Solution:

$$\begin{aligned}\mathrm{P}_{\text{out}} &= \mathrm{P}(\{P_{r,1} < P_{\text{thresh}}\} \cap \{P_{r,2} < P_{\text{thresh}}\}) = \mathrm{P}(P_{r,1} < P_{\text{thresh}})\mathrm{P}(P_{r,2} < P_{\text{thresh}}) \\ &= \mathrm{P}\left(Z_1 < P_{\text{thresh}} - A_t - B\log(D/2)\right)\mathrm{P}\left(Z_2 < P_{\text{thresh}} - A_t - B\log(D/2)\right) \\ &= \mathrm{P}\left(Z_1 < -\Delta\right)\mathrm{P}\left(Z_2 < -\Delta\right) = \left[Q\left(\frac{\Delta}{\sigma} \right) \right]^2\end{aligned}$$

(b) Now suppose Z_1 and Z_2 are correlated in the following way.

$$Z_1 = aY_1 + bY \quad \text{and} \quad Z_2 = aY_2 + bY,$$

where Y, Y_1 and Y_2 are *independent* $\mathcal{N}(0, \sigma^2)$ random variables, and a and b are such that $a^2 + b^2 = 1$.

Show that

$$\mathrm{P}_{\text{out}} = \int_{-\infty}^{\infty} \left[Q\left(\frac{\Delta + by\sigma}{a\sigma} \right) \right]^2 \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

Solution:

$$\begin{aligned}\mathrm{P}_{\text{out}} &= \mathrm{P}(\{aY_1 + bY < -\Delta\} \cap \{aY_2 + bY < -\Delta\}) \\ &= \int_{-\infty}^{\infty} \mathrm{P}(\{aY_1 + by < -\Delta\} \cap \{aY_2 + by < -\Delta\}) p_Y(y) dy \\ &= \int_{-\infty}^{\infty} \mathrm{P}(aY_1 < -\Delta - by)\mathrm{P}(aY_2 < -\Delta - by) p_Y(y) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \left(Q\left(\frac{\Delta + by}{a\sigma} \right) \right)^2 e^{-\frac{y^2}{2\sigma^2}} dy = \int_{-\infty}^{\infty} \left(Q\left(\frac{\Delta + bw\sigma}{a\sigma} \right) \right)^2 \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw\end{aligned}$$

which is the desired expression.

(c) Compare the outage probabilities of (i) and (ii) for the special case of $a = b = 1/\sqrt{2}$, $\sigma = 8$ and $\Delta = 5$. (Use a numerical integration routine for P_{out} of (ii).)

Solution:

$$\begin{aligned}\mathrm{P}_{\text{out},1} &= 0.0707 \\ \mathrm{P}_{\text{out},2} &= 0.1316\end{aligned} \tag{2}$$

Thus, correlation reduces diversity gain.

7. *Squared-envelope correlation for isotropic fading:*

(a) Prove that the squared-envelope covariance function for a *Rayleigh* fading process $\{E(t)\}$ is given by:

$$C_{\alpha^2}(\tau) = |R_E(\tau)|^2$$

Hint: It may be easier to first show that $R_{\alpha^2}(\tau) = |R_E(\tau)|^2 + 1$. You may find the following result to be useful: if X_1, X_2, X_3, X_4 are jointly Gaussian *zero-mean* random variables, then

$$\mathrm{E}[X_1X_2X_3X_4] = \mathrm{E}[X_1X_2]\mathrm{E}[X_3X_4] + \mathrm{E}[X_1X_3]\mathrm{E}[X_2X_4] + \mathrm{E}[X_1X_4]\mathrm{E}[X_2X_3].$$

Solution: In particular, when $X_1 = X_2$ and $X_3 = X_4$ we get

$$\mathrm{E}[X_1X_1X_3X_3] = \mathrm{E}[X_1X_1]\mathrm{E}[X_3X_3] + 2\left(\mathrm{E}[X_1X_3]\right)^2.$$

Thus

$$\begin{aligned}
R_{\alpha^2}(\tau) &= \mathrm{E}\left[\left(E_I^2(t+\tau) + E_Q^2(t+\tau)\right)\left(E_I^2(t) + E_Q^2(t)\right)\right] \\
&= \mathrm{E}[E_I^2(t+\tau)E_I^2(t)] + \mathrm{E}[E_Q^2(t+\tau)E_Q^2(t)] + \mathrm{E}[E_I^2(t+\tau)E_Q^2(t)] + \mathrm{E}[E_Q^2(t+\tau)E_I^2(t)] \\
&= 2\mathrm{E}[E_I^2(t+\tau)E_I^2(t)] + 2\mathrm{E}[E_I^2(t+\tau)E_Q^2(t)] \\
&= 2\left(\mathrm{E}[E_I^2(t+\tau)]\mathrm{E}[E_I^2(t)] + 2\left(\mathrm{E}[E_I(t+\tau)E_I(t)]\right)^2\right) \\
&\quad + 2\left(\mathrm{E}[E_I^2(t+\tau)]\mathrm{E}[E_Q^2(t)] + 2\left(\mathrm{E}[E_I(t+\tau)E_Q(t)]\right)^2\right) \\
&= 2\left(\frac{1}{4} + 2(R_{E_I}(\tau))^2\right) + 2\left(\frac{1}{4} + 2(R_{E_IE_Q}(\tau))^2\right) \\
&= 1 + 4\left((R_{E_I}(\tau))^2 + (R_{E_IE_Q}(\tau))^2\right) = 1 + |R_E(\tau)|^2
\end{aligned}$$

(b) Prove that the squared-envelop covariance function for a *Ricean* fading process $\{E(t)\}$ is:

$$C_{\alpha^2}(\tau) = \left(\frac{1}{\kappa+1}\right)^2\left[|R_{\check{E}}(\tau)|^2 + 2\kappa\mathrm{Re}\left[R_{\check{E}}(\tau)e^{-j2\pi f_{\max}\tau\cos\theta_0}\right]\right],$$

where κ is the Rice factor and θ_0 is the angle of arrival of the specular component.

Solution:

$$\begin{aligned}
R_{\alpha^2}(\tau) &= \mathrm{E}\left[\alpha^2(t)\alpha^2(t+\tau)\right] = \mathrm{E}\left[E(t)E^*(t)E(t+\tau)E^*(t+\tau)\right] \\
&= \mathrm{E}[\left(\beta_0 e^{j\phi_0(t)} + \sqrt{1-\beta_0^2}\check{E}(t)\right)\left(\beta_0 e^{-j\phi_0(t)} + \sqrt{1-\beta_0^2}\check{E}^*(t)\right) \\
&\quad \left(\beta_0 e^{j\phi_0(t+\tau)} + \sqrt{1-\beta_0^2}\check{E}(t+\tau)\right)\left(\beta_0 e^{-j\phi_0(t+\tau)} + \sqrt{1-\beta_0^2}\check{E}^*(t+\tau)\right)] \\
&\overset{a}{=} \beta_0^4 + (1-\beta_0^2)^2 R_{\check{\alpha}^2\check{\alpha}^2}(\tau) + \beta_0^2(1-\beta_0^2)E[\check{\alpha}^2(t)] + \beta_0^2(1-\beta_0^2)E[\check{\alpha}^2(t+\tau)] \\
&\quad \beta_0^2(1-\beta_0^2)e^{j(\phi_0(t)-\phi_0(t+\tau))}\mathrm{E}[\check{E}^*(t)\check{E}(t+\tau)] + \beta_0^2(1-\beta_0^2)e^{-j(\phi_0(t)-\phi_0(t+\tau))}\mathrm{E}[\check{E}(t)\check{E}^*(t+\tau)] \\
&= \beta_0^4 + (1-\beta_0^2)^2(1 + |R_{\check{E}}(\tau)|^2) + 2\beta_0^2(1-\beta_0^2) \\
&\quad + 2\beta_0^2(1-\beta_0^2)\mathrm{Re}\left[\mathrm{E}[\check{E}^*(t)\check{E}(t+\tau)]e^{-j(\phi_0(t+\tau)-\phi_0(t))}\right] \\
&= 1 + (1-\beta_0^2)^2|R_{\check{E}}(\tau)|^2 + 2\beta_0^2(1-\beta_0^2)\mathrm{Re}\left[\mathrm{E}[\check{E}^*(t)\check{E}(t+\tau)]e^{-j(\phi_0(t+\tau)-\phi_0(t))}\right] \\
&= 1 + \left(\frac{1}{1+\kappa}\right)^2|R_{\check{E}}(\tau)|^2 + 2\left(\frac{1}{1+\kappa}\right)^2\kappa\mathrm{Re}\left[R_{\check{E}}(\tau)e^{-j2\pi f_{\max}\tau\cos\theta_0}\right]
\end{aligned}$$

Note that there are 16 terms in the expectation. Of these 16 terms, 8 involve only one phase term ϕ_0. Since the specular component is independent of the diffuse component, and the marginal distribution of the phase is uniform, these eight terms go to zero. Of the six terms containing two phase terms, two vanish because they involve phase of the form $\phi_0(t) + \phi_0(t+\tau)$. Thus, 6 terms are left in a.

$$C_{\alpha^2} = R_{\alpha^2} - \left(\mathrm{E}[\alpha^2(t)]\right)^2 = \left(\frac{1}{1+\kappa}\right)^2 \left[|R_{\check{E}}(\tau)|^2 + 2\kappa \mathrm{Re}\left[R_{\check{E}}(\tau)e^{-j2\pi f_{\max}\tau\cos\theta_0}\right]\right]$$

8. *Power Spectrum of Rayleing Fading Process.* Consider a Rayleigh fading environment with angular power density $p(\theta)$.

   (a) Show that:

   $$S_E(f) = \int_0^{\pi} [p(\theta) + p(-\theta)]\delta(f - f_{\max}\cos\theta)d\theta$$

   (b) Now show that $S_E(f)$ has the closed-form expression:

   $$S_E(f) = \begin{cases} \dfrac{p(\cos^{-1}(f/f_{\max})) + p(-\cos^{-1}(f/f_{\max}))}{\sqrt{f_{\max}^2 - f^2}} & \text{for } |f| < f_{\max} \\ 0 & \text{otherwise} \end{cases}$$

   (c) Specialize this result for isotropic Rayleigh fading, i.e., $p(\theta) = 1/2\pi$.

Solution:

$$\begin{aligned} S_E(f) &= \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} p(\theta)e^{j2\pi f_{\max}\tau\cos\theta}e^{-j2\pi f\tau}d\theta d\tau = \int_{-\pi}^{\pi} p(\theta)\int_{-\infty}^{\infty} e^{-j2\pi(f - f_{\max}\tau\cos\theta)}d\tau d\theta \\ &= \int_{-\pi}^{\pi} p(\theta)\delta(f - f_{\max}\cos\theta)d\theta = \int_0^{\pi}\left[p(\theta) + p(-\theta)\right]\delta(f - f_{\max}\cos\theta)d\theta \\ &= \int_{-f_{\max}}^{f_{\max}}\left[p(\cos^{-1}(u/f_{\max})) + p(-\cos^{-1}(u/f_{\max}))\right]\delta(u-f)\frac{du}{f_{\max}\sqrt{1-(u/f_{\max})^2}} \\ &= \begin{cases} \frac{\left[p(\cos^{-1}(f/f_{\max})) + p(-\cos^{-1}(f/f_{\max}))\right]}{\sqrt{f_{\max}^2 - f^2}} & \text{if } |f| < f_{\max}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$