HOMEWORK ASSIGNMENT 2 (due October 3)

1. *Estimating the squared-envelope using pilot symbols*
   Consider a digital communication system over an *isotropic* flat Rayleigh fading channel with maximum Doppler frequency $\nu_{\max}$. To improve the detection of the transmitted symbols, the receiver attempts to estimate the squared-envelope process $\beta^2(t)$ every symbol period T_s. Define the discrete process X_k by
   $$X_k = \beta^2(kT_s), \;\; k = 0, 1, \ldots,$$
   To aid the estimation of the process X_k, the system transmits a pilot symbol (that is known a priori to the receiver) every M symbols. Using these pilot symbols, the receiver determines exactly the values X_0, X_M, $X_{2M}, \ldots$. Now the receiver needs to interpolate between the known X values using knowledge of the autocorrelation of the squared-envelope process $R_{\beta^2}(\xi)$.

   (a) For $k = 1, 2, \ldots, M-1$, consider the estimation of X_k based on X_0 and X_M. Determine equations for the optimum weights a_k, b_k for the LMMSE estimate $\hat{X}_k$:
   $$\hat{X}_k = a_k(X_0 - 1) + b_k(X_M - 1) + 1$$
   for $k = 1, \cdots, M-1$. (Hint: Use the orthogonality principle for each k.)

   (b) Using Matlab, find the optimum tap weights for the case where $M = 10$, $\nu_{\max} = 10Hz$ and $T_s = \frac{1}{100\pi}$ seconds.

2. *Derivative of Envelope Process.*

   Hint: If $\{X(t)\}$ is a real-valued WSS process with $Y(t) = \frac{d}{dt}X(t)$, then
   $$R_{XY}(\tau) = -\frac{d}{d\tau}R_X(\tau) \, , \text{ and } \; R_Y(\tau) = -\frac{d^2}{d\tau^2}R_X(\tau) \, .$$

   Consider an isotropic Rayleigh fading process $E(t)$ with envelope $\beta(t)$ and phase $\phi(t)$.

   (a) Show that the derivative of the envelope process is given by
   $$\dot{\beta}(t) = \frac{d}{dt}\beta(t) = \dot{E}_I(t)\,\cos\phi(t) + \dot{E}_Q(t)\,\sin\phi(t)$$
   where $\dot{E}_I(t)$ and $\dot{E}_Q(t)$ are the derivatives of the in-phase and quadrature processes, respectively.

   (b) Show that $\dot{E}_I(t)$ and $\dot{E}_Q(t)$ are *mutually independent, zero mean, Gaussian* processes with
   $$R_{\dot{E}_I}(\xi) = R_{\dot{E}_Q}(\xi) = \pi\,\nu_{\max}^2 \int_{-\pi}^{\pi} \cos(2\pi\nu_{\max}\xi\cos\theta)\,\cos^2\theta\,d\theta \, .$$

   (c) For fixed t, show that $\dot{\beta}(t)$ is a Gaussian random variable with zero mean and variance $\pi^2\nu_{\max}^2$. *Hint*: Condition on $\phi(t)$ first.

   (d) For fixed t, show that $\beta(t)$ and $\dot{\beta}(t)$ are independent random variables, and write down their joint pdf.

3. *Block Correlated Fading Model.* As we discussed in class (also see page 58 of the notes), the block (independent) fading model can be generalized to allow for both time-selectivity within the block ($TB_b > 1$) as well as correlation across blocks. For simplicity consider the frequency-flat case, and derive a reasonable model for correlation across blocks that reflects the large scale variations in the channel. You may assume that the median of the large scale variations is always normalized to 1 via power control.

4. *Useful results.* Show that

   (a)
$$\int_0^\infty Q(\sqrt{x})\frac{e^{-x/\gamma}}{\gamma}dx = \frac{1}{2}\left(1-\sqrt{\frac{\gamma}{2+\gamma}}\right).$$

   (b) For $t > 0$,
$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta.$$

   This is an alternative, and useful form, for the $Q(\cdot)$ function.

5. *MPSK in Rayleigh fading.*

   (a) For MPSK signaling,
$$\mathrm{P_e}(\gamma_\mathrm{s}) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left[-\frac{\gamma_\mathrm{s}\sin^2(\pi/M)}{\sin^2\theta}\right]d\theta\ .$$

   Using this expression how that the average symbol error probability $\overline{\mathrm{P}}_\mathrm{e}$ for MPSK signaling in Rayleigh fading is given in closed form by
$$\overline{\mathrm{P}}_\mathrm{e} = \left(1-\frac{1}{M}\right) - \frac{1}{\sqrt{1+a^2}}\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{\cot\pi/M}{\sqrt{1+a^2}}\right)\right],$$

   where $a^2 = \frac{1}{\overline{\gamma}_\mathrm{s}\sin^2\pi/M}$.

   Hint: You may need to use the following integral
$$\int_{\theta_1}^{\theta_2}\frac{1}{\mathrm{cosec}^2\theta + a^2}d\theta = \frac{1}{a^2}\left[\frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{\cot\theta}{\sqrt{1+a^2}}\right)-\left(\frac{\pi}{2}-\theta\right)\right]_{\theta_1}^{\theta_2} \quad \text{for } 0 \le \theta_1 \le \theta_2 \le \pi/2.$$

6. *Non-coherent M-ary orthogonal signaling in Ricean fading*:

   (a) For Non-coherent M-ary orthogonal signaling,
$$\mathrm{P_e}(\gamma_\mathrm{s}) = \sum_{n=1}^{M-1}(-1)^{n+1}\binom{M-1}{n}\frac{1}{n+1}\exp\left[-\frac{n\gamma_\mathrm{s}}{(n+1)}\right].$$

   Using this expression, show that $\overline{\mathrm{P}}_\mathrm{e}$ under Ricean fading with Rice factor κ is given by
$$\overline{\mathrm{P}}_\mathrm{e} = \sum_{n=1}^{M-1}(-1)^{n+1}\binom{M-1}{n}\frac{\tilde{p}(n,\overline{\gamma}_\mathrm{s})}{n+1},$$

   where
$$\tilde{p}(n,\overline{\gamma}_\mathrm{s}) = \frac{\kappa+1}{\kappa+1+\frac{n}{n+1}\,\overline{\gamma}_\mathrm{s}}\exp\left(-\frac{\frac{n}{n+1}\,\overline{\gamma}_\mathrm{s}\,\kappa}{\kappa+1+\frac{n}{n+1}\,\overline{\gamma}_\mathrm{s}}\right)$$

7. *Nakagami-m fading.* The first order statistics of a flat fading channel are sometimes approximated by a pdf from the Nakagami-m family:

$$p_\beta(x) = \frac{2m^m x^{2m-1}}{\Gamma(m) a^m} \exp\left(-\frac{mx^2}{a}\right) 1\!\!1_{\{x>0\}}, \quad m > 0.5$$

where $a = \mathrm{E}[\beta^2] = 1$, and $\Gamma(\cdot)$ is the Gamma function which is defined by the integral $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, for $x > 0$. (Properties of $\Gamma(\cdot)$ include: $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(0.5) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(n+1) = n!$, for positive integer n.)

Note: If m is positive integer, which means that $\Gamma(m) = (m-1)!$, the above p.d.f. is a central chi-squared distribution with $2m$ degrees of freedom, which is also the p.d.f. of the sum of m independent and identically distributed exponential random variables – sometimes called an Erlang distribution.

(a) Show that the pdf of $\gamma_\mathsf{b} = \beta^2 \overline{\gamma}_\mathsf{b}$ is given by

$$p_{\gamma_\mathsf{b}}(x) = \left(\frac{m}{\overline{\gamma}_\mathsf{b}}\right)^m \frac{x^{m-1}}{\Gamma(m)} \exp\left(-\frac{mx}{\overline{\gamma}_\mathsf{b}}\right) 1\!\!1_{\{x>0\}}.$$

(b) If m is a positive integer, show that the c.d.f. of γ_b is given by:

$$F_{\gamma_\mathsf{b}}(x) = \int_0^x p_{\gamma_\mathsf{b}}(y) dy = 1 - \sum_{i=0}^{m-1} \left(\frac{m}{\overline{\gamma}_\mathsf{b}}\right)^i \frac{x^i}{i!} \exp\left(-\frac{mx}{\overline{\gamma}_\mathsf{b}}\right).$$

(c) Now show that if m is a positive integer, $\overline{\mathsf{P}}_\mathsf{b}$ for BPSK signaling in slow, flat Nakagami-m fading is given by

$$\overline{\mathsf{P}}_\mathsf{b} = \frac{1}{2} - \frac{\tilde{\sigma}}{2} \sum_{i=0}^{m-1} \left(\frac{m\tilde{\sigma}^2}{4\overline{\gamma}_\mathsf{b}}\right)^i \frac{(2i)!}{(i!)^2},$$

where $\tilde{\sigma} = \sqrt{\overline{\gamma}_\mathsf{b}/(\overline{\gamma}_\mathsf{b} + m)}$.

Hint: You may want to use the fact that the even moments of a $\mathcal{N}(0,1)$ random variable X are given by:

$$\mathrm{E}[X^{2i}] = \frac{(2i)!}{i!\, 2^i}.$$