SOLUTIONS TO HOMEWORK ASSIGNMENT 2

1. Estimating the squared-envelope using pilot symbols

Consider a digital communication system over an isotropic flat Rayleigh fading channel with maximum Doppler frequency $\nu_{\text{max}}$. To improve the detection of the transmitted symbols, the receiver attempts to estimate the squared-envelope process $\beta^2(t)$ every symbol period $T_s$. Define the discrete process $X_k$ by

$$X_k = \beta^2(kT_s), \quad k = 0, 1, \ldots,$$

To aid the estimation of the process $X_k$, the system transmits a pilot symbol (that is known a priori to the receiver) every $M$ symbols. Using these pilot symbols, the receiver determines exactly the values $X_0, X_M, X_{2M}, \ldots$. Now the receiver needs to interpolate between the known $X$ values using knowledge of the autocorrelation of the squared-envelope process $R_{\beta^2}(\xi)$.

(a) For $k = 1, 2, \ldots, M - 1$, consider the estimation of $X_k$ based on $X_0$ and $X_M$. Determine equations for the optimum weights $a_k, b_k$ for the LMMSE estimate $\hat{X}_k$:

$$\hat{X}_k = a_k(X_0 - 1) + b_k(X_M - 1) + 1$$

for $k = 1, \ldots, M - 1$. (Hint: Use the orthogonality principle for each $k$.)

Solution: Let $C[m] = \mathbb{E}[(X_j - 1)(X_{j+m} - 1)]$. It is clear that

$$C[m] = J_0^2(2\pi\nu_{\text{max}}mT_s).$$

Note that $\mathbb{E}[\hat{X}_k] = 1 = \mathbb{E}[X_k]$ since $\mathbb{E}[X_0 - 1] = \mathbb{E}[X_M - 1] = 0$. Applying the orthogonality principle, we have $\mathbb{E}[(X_k - \hat{X}_k)X_0] = \mathbb{E}[(X_k - \hat{X}_k)X_M] = 0$. Thus,

$$\mathbb{E}[(Y_k - a_kX_0 - b_kX_M)Y_0] = 0$$

and

$$\mathbb{E}[(Y_k - a_kX_0 - b_kX_M)Y_M] = 0$$

where $Y_k = X_k - 1$. This implies $C[k] = a_kC[0] + b_kC[M]$ and $C[M - k] = a_kC[M] + b_kC[0]$. Solving for $a_k$ and $b_k$, we get

$$a_k = \frac{C[k]C[0] - C[M]C[M - k]}{(C[0])^2 - (C[M])^2},$$

$$b_k = \frac{C[M - k]C[0] - C[k]C[M]}{(C[0])^2 - (C[M])^2}.$$

(b) Using Matlab, find the optimum tap weights for the case where $M = 10$, $\nu_{\text{max}} = 10Hz$ and $T_s = \frac{1}{1000\pi}$ seconds.
**Solution:** The taps are given by:

\[
\begin{array}{cccc}
k & a_k & b_k \\
1 & 0.9768 & 0.0666 \\
2 & 0.9143 & 0.1616 \\
3 & 0.8177 & 0.2803 \\
4 & 0.6954 & 0.4156 \\
5 & 0.5576 & 0.5576 \\
6 & 0.4156 & 0.6954 \\
7 & 0.2803 & 0.8177 \\
8 & 0.1616 & 0.9143 \\
9 & 0.0666 & 0.9768 \\
\end{array}
\]


**Hint:** If \( \{X(t)\} \) is a real-valued WSS process with \( Y(t) = \frac{d}{dt}X(t) \), then

\[
R_{XY}(\tau) = -\frac{d}{d\tau}R_X(\tau) \quad \text{and} \quad R_Y(\tau) = -\frac{d^2}{d\tau^2}R_X(\tau) .
\]

Consider an isotropic Rayleigh fading process \( E(t) \) with envelope \( \beta(t) \) and phase \( \phi(t) \).

(a) Show that the derivative of the envelope process is given by

\[
\dot{\beta}(t) = \frac{d}{dt} \beta(t) = \dot{E}_I(t) \cos \phi(t) + \dot{E}_Q(t) \sin \phi(t)
\]

where \( \dot{E}_I(t) \) and \( \dot{E}_Q(t) \) are the derivatives of the in-phase and quadrature processes, respectively.

(b) Show that \( \dot{E}_I(t) \) and \( \dot{E}_Q(t) \) are mutually independent, zero mean, Gaussian processes with

\[
R_{\dot{E}_I}(\xi) = R_{\dot{E}_Q}(\xi) = \pi \nu_{\max}^2 \int_{-\pi}^{\pi} \cos(2\pi \nu_{\max} \xi \cos \theta) \cos^2 \theta \, d\theta .
\]

(c) For fixed \( t \), show that \( \dot{\beta}(t) \) is a Gaussian random variable with zero mean and variance \( \pi^2 \nu_{\max}^2 \).

**Hint:** Condition on \( \phi(t) \) first.

(d) For fixed \( t \), show that \( \beta(t) \) and \( \dot{\beta}(t) \) are independent random variables, and write down their joint pdf.

**Solution:** We talked about this in class.

3. Block Correlated Fading Model. As we discussed in class (also see page 58 of the notes), the block (independent) fading model can be generalized to allow for both time-selectivity within the block \( (TB_b > 1) \) as well as correlation across blocks. For simplicity consider the frequency-flat case, and derive a reasonable model for correlation across blocks that reflects the large scale variations in the channel. You may assume that the median of the large scale variations is always normalized to 1 via power control.

**Solution:** See Y. Liang and V.V. Veeravalli. “Capacity of Noncoherent Time-Selective Block Fading Channels.” IEEE Transactions on Information Theory, 50(12):3095-3110, December 2004, for one way to model the correlation across blocks.
4. Useful results. Show that

(a) \[ \int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \frac{1}{2} \left( 1 - \sqrt{\frac{\gamma}{2 + \gamma}} \right). \]

Solution:

\[ \int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \int_0^\infty \int_\sqrt{x}^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{-x/\gamma} \frac{1}{\gamma} dydx = \int_0^\infty \int_0^y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} e^{-x/\gamma} dx dy \]

\[ = \int_0^\infty \frac{1}{\sqrt{2\pi}} (1 - e^{-x^2/2}) e^{-x/\gamma} dy \]

\[ = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dy - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-x/\gamma} dy = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{\gamma}{2 + \gamma}} \]

(b) For \( t > 0 \),

\[ Q(t) = \frac{1}{\pi} \int_0^{\pi/2} \exp \left( - \frac{x^2}{2 \sin^2 \theta} \right) d\theta. \]

This is an alternative, and useful form, for the \( Q(\cdot) \) function.

Solution:

\[ Q(t) = \frac{1}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dx = \frac{1}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dxdy = \frac{1}{\pi} \int_0^{\pi/2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} rdrd\theta \]

\[ = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-t^2/2 \cos^2 \theta} d\theta = \frac{1}{\pi} \int_0^{\pi/2} e^{-t^2/2 \cos^2 \theta} d\theta = \frac{1}{\pi} \int_0^{\pi/2} e^{-t^2/2 \sin^2 \theta} d\theta \]

5. MPSK in Rayleigh fading.

For MPSK signaling,

\[ P_e(\gamma_s) = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \exp \left[ -\frac{\gamma_s \sin^2(\pi/M)}{\sin^2 \theta} \right] d\theta. \]

Using this expression how that the average symbol error probability \( P_e \) for MPSK signaling in Rayleigh fading is given in closed form by

\[ \overline{P_e} = \left( 1 - \frac{1}{M} \right) - \frac{1}{\sqrt{1 + a^2}} \left[ \frac{1}{2} + \frac{1}{\pi} \left( \cot \pi/M \right) \right], \]

where \( a^2 = \frac{\gamma_s}{\sin^2 \pi/M} \).

Hint: You may need to use the following integral

\[ \int_{\theta_1}^{\theta_2} \frac{1}{\cosec^2 \theta + a^2} d\theta = \frac{1}{a^2} \left[ \frac{1}{\sqrt{1 + a^2}} \tan^{-1} \left( \frac{\cot \theta}{\sqrt{1 + a^2}} \right) - \left( \frac{\pi}{2} - \theta \right) \right]_{\theta_1}^{\theta_2} \text{ for } 0 \leq \theta_1 \leq \theta_2 \leq \pi/2. \]
Solution:

\[
P_e = \int_0^\infty P_e(x) p_{\gamma_s}(x) dx = \frac{1}{\pi} \int_0^\infty \int_0^{\pi - \frac{\pi}{M}} e^{-\frac{x \sin^2 \frac{\pi}{M}}{\sin^2 \theta}} \left(\frac{1}{\gamma_s} e^{-\frac{x}{\gamma_s}}\right) d\theta \frac{1}{\gamma_s} e^{-\frac{x}{\gamma_s}} dx
\]

\[
= \frac{1}{\pi} \int_0^{\pi - \frac{\pi}{M}} \frac{1}{\gamma_s} \int_0^\infty e^{-x\left(\frac{\sin^2 \frac{\pi}{M}}{\sin^2 \theta} + \frac{1}{\kappa}\right)} dx d\theta = \frac{1}{\pi \gamma_s} \int_0^{\pi - \frac{\pi}{M}} \frac{\sin^2 \frac{\pi}{M}}{\sin^2 \theta} + \frac{1}{\kappa} d\theta
\]

\[
= \frac{1}{\pi \gamma_s \sin^2 \frac{\pi}{M}} \int_0^{(M-1)\pi} \frac{1}{\csc^2 \theta + \frac{1}{\kappa \sin^2 \frac{\pi}{M}}} \frac{1}{\csc^2 \theta + a^2} d\theta = \frac{a^2}{\pi} \int_0^{(M-1)\pi} \frac{1}{\csc^2 \theta + \frac{1}{\kappa}} d\theta
\]

\[
= \frac{1}{\pi} \left[ \frac{1}{\sqrt{1 + a^2}} \tan^{-1}\left(\frac{1}{\sqrt{1 + a^2}}\right) - \frac{\pi}{2\sqrt{1 + a^2}} + \frac{\pi}{M} \right]
\]

\[
= \left(1 - \frac{1}{M}\right) - \frac{1}{\sqrt{1 + a^2}} \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{\sqrt{1 + a^2}}{\sqrt{1 + a^2}}\right) \right]
\]

where \( a^{-1} = \sqrt{\gamma_s \sin \frac{\pi}{M}} \).

6. Non-coherent M-ary orthogonal signaling in Ricean fading:

For Non-coherent M-ary orthogonal signaling,

\[
P_e(\gamma_s) = \sum_{n=1}^{M-1} (-1)^{n+1} \left(\frac{M - 1}{n}\right) \frac{1}{n + 1} \exp \left[ -\left(\frac{n \gamma_s}{(n + 1)}\right)\right].
\]

Using this expression, show that \( P_e \) under Ricean fading with Rice factor \( \kappa \) is given by

\[
P_e = \sum_{n=1}^{M-1} (-1)^{n+1} \left(\frac{M - 1}{n}\right) \frac{1}{n + 1} \tilde{p}(n, \gamma_s)
\]

where

\[
\tilde{p}(n, \gamma_s) = \frac{\kappa + 1}{\kappa + 1 + \frac{n}{n + 1} \gamma_s} \exp \left( -\left(\frac{n \gamma_s}{\kappa + 1 + \frac{n}{n + 1} \gamma_s}\right)\right)
\]

Solution:

\[
P_e = \int_0^\infty P_e(x) p_{\gamma_s}(x) dx
\]

For Ricean fading with Rice factor \( \kappa \),

\[
p_{\gamma_s}(x) = \frac{\kappa + 1}{\gamma_s} I_0 \left(2 \sqrt{\frac{\kappa (\kappa + 1)x}{\gamma_s}}\right) e^{-\kappa - \frac{(\kappa + 1)x}{\gamma_s}} \mathbb{I}_{\{x \geq 0\}}
\]

Now,

\[
\int_0^\infty e^{-\frac{n x}{\kappa + 1 + \frac{n}{n + 1} \gamma_s}} p_{\gamma_s}(x) dx = \frac{\kappa + 1}{\kappa + 1 + \frac{n}{n + 1} \gamma_s} e^{-\frac{n \gamma_s}{\kappa + 1 + \frac{n}{n + 1} \gamma_s}} = \tilde{p}(n, \gamma)
\]

This follows by using the fact that a Ricean PDF must integrate to 1. The result follows.
7. Nakagami-

fading. The first order statistics of a flat fading channel are sometimes approximated by a pdf from the Nakagami-

family:

\[ p_{\beta}(x) = \frac{2m^m x^{2m-1}}{\Gamma(m) a^m} \exp \left( -\frac{m x^2}{a} \right) \mathbb{1}_{\{x>0\}}, \quad m > 0.5 \]

where \( a = \mathbb{E}[\beta^2] = 1 \), and \( \Gamma(\cdot) \) is the Gamma function which is defined by the integral \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \), for \( x > 0 \). (Properties of \( \Gamma(\cdot) \) include: \( \Gamma(x+1) = x\Gamma(x) \), \( \Gamma(0.5) = \sqrt{\pi} \), \( \Gamma(1) = 1 \), and \( \Gamma(n+1) = n! \), for positive integer \( n \).)

Note: If \( m \) is a positive integer, which means that \( \Gamma(m) = (m-1)! \), the above p.d.f. is a central chi-squared distribution with \( 2m \) degrees of freedom, which is also the p.d.f. of the sum of \( m \) independent and identically distributed exponential random variables – sometimes called an Erlang distribution.

(a) Show that the pdf of \( \gamma_b = \beta^2 \gamma_b \) is given by

\[ p_{\gamma_b}(x) = \left( \frac{m}{\gamma_b} \right)^m \frac{x^{m-1}}{\Gamma(m)} \exp \left( -\frac{m x}{\gamma_b} \right) \mathbb{1}_{\{x>0\}}. \]

Solution:

\[ p_{\gamma_b}(x) = \frac{1}{2\sqrt{x/\gamma_b}} p_{\beta}(\sqrt{x/\gamma_b}) = \frac{1}{2\sqrt{x/\gamma_b}} \frac{2m^m (\sqrt{x/\gamma_b})^{2m-1}}{\Gamma(m)} \exp \left( -\frac{m x}{\gamma_b} \right) \mathbb{1}_{\{x>0\}}. \]

(b) If \( m \) is a positive integer, show that the c.d.f. of \( \gamma_b \) is given by:

\[ F_{\gamma_b}(x) = \int_0^x p_{\gamma_b}(y) \, dy = 1 - \sum_{i=0}^{m-1} \left( \frac{m}{\gamma_b} \right)^i \frac{x^i}{i!} \exp \left( -\frac{m x}{\gamma_b} \right). \]

Solution: When \( m \) is a positive integer, \( \Gamma(m) = (m-1)! \),

\[ F_{\gamma_b}(y) = \int_0^y \left( \frac{m}{\gamma_b} \right)^m \frac{x^{m-1}}{(m-1)!} \exp \left( -\frac{m x}{\gamma_b} \right) \, dx \]

Integrating by parts \( m \) times, we get the result.

(c) Now show that if \( m \) is a positive integer, \( \bar{P}_b \) for BPSK signaling in slow, flat Nakagami-

fading is given by

\[ \bar{P}_b = \frac{1}{2} - \frac{\bar{\sigma}}{2} \sum_{i=0}^{m-1} \left( \frac{m \bar{\sigma}^2}{4 \gamma_b} \right)^i \frac{(2i)!}{(i!)^2}, \]

where \( \bar{\sigma} = \sqrt{\gamma_b/(\gamma_b + m)} \).

Hint: You may want to use the fact that the even moments of a \( \mathcal{N}(0,1) \) random variable \( X \) are given by:

\[ \mathbb{E}[X^{2i}] = \frac{(2i)!}{i! 2^i}. \]

Solution:
\[ P_b = \int_0^\infty Q(\sqrt{2x}) p_{\gamma b}(x) dx = \int_0^\infty p_{\gamma b}(x) \left( \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right) dx \]

\[ = \int_0^\infty \int_0^{\frac{y^2}{x}} p_{\gamma b}(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dxdy = \int_0^\infty F_{\gamma b} \left( \frac{y^2}{2} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \]

\[ = \frac{1}{2} - \sum_{i=0}^{m-1} \left( \frac{m}{2\gamma b} \right)^i \frac{1}{i!} \int_0^\infty y^{2i} e^{-\frac{y^2}{2\gamma b} \left( \frac{m}{b} + 1 \right)} dy \]

Now,

\[ \int_0^\infty y^{2i} e^{-\frac{y^2}{2\gamma b} \left( \frac{m}{b} + 1 \right)} dy = \sigma \int_0^\infty \sigma^{2i} u^{2i} e^{-\frac{u^2}{2\gamma b}} du = \frac{1}{2} \sigma^{2i+1} (2i)! (i!)^2 \]

from which the final result follows.