October 20, 2007

SOLUTIONS TO HOMEWORK ASSIGNMENT 2

1. *Estimating the squared-envelope using pilot symbols*
Consider a digital communication system over an *isotropic* flat Rayleigh fading channel with maximum Doppler frequency $\nu_{\max}$. To improve the detection of the transmitted symbols, the receiver attempts to estimate the squared-envelope process $\beta^2(t)$ every symbol period T_s. Define the discrete process X_k by
$$X_k = \beta^2(kT_s), \;\; k = 0, 1, \ldots,$$
To aid the estimation of the process X_k, the system transmits a pilot symbol (that is known a priori to the receiver) every M symbols. Using these pilot symbols, the receiver determines exactly the values X_0, X_M, $X_{2M}, \ldots$. Now the receiver needs to interpolate between the known X values using knowledge of the autocorrelation of the squared-envelope process $R_{\beta^2}(\xi)$.

   (a) For $k = 1, 2, \ldots, M-1$, consider the estimation of X_k based on X_0 and X_M. Determine equations for the optimum weights a_k, b_k for the LMMSE estimate $\hat{X}_k$:
$$\hat{X}_k = a_k(X_0 - 1) + b_k(X_M - 1) + 1$$
for $k = 1, \cdots, M-1$. (Hint: Use the orthogonality principle for each k.)
Solution: Let $C[m] = \mathrm{E}[(X_j - 1)(X_{j+m} - 1)]$. It is clear that
$$C[m] = J_0^2(2\pi\nu_{\max} m T_s).$$
Note that $\mathrm{E}[\hat{X}_k] = 1 = \mathrm{E}[X_k]$ since $\mathrm{E}[X_0 - 1] = \mathrm{E}[X_M - 1] = 0$. Applying the orthogonality principle, we have $\mathrm{E}[(X_k - \hat{X}_k)X_0] = E[(X_k - \hat{X}_k)X_M] = 0$. Thus,
$$\mathrm{E}\left[(Y_k - a_k Y_0 - b_k Y_M)Y_0\right] = 0$$
and
$$\mathrm{E}\left[(Y_k - a_k Y_0 - b_k Y_M)Y_M\right] = 0$$
where $Y_k = X_k - 1$. This implies $C[k] = a_k C[0] + b_k C[M]$ and $C[M-k] = a_k C[M] + b_k C[0]$. Solving for a_k and b_k, we get
$$a_k = \frac{C[k]C[0] - C[M]C[M-k]}{(C[0])^2 - (C[M])^2},$$
$$b_k = \frac{C[M-k]C[0] - C[k]C[M]}{(C[0])^2 - (C[M])^2}.$$

   (b) Using Matlab, find the optimum tap weights for the case where $M = 10$, $\nu_{\max} = 10Hz$ and $T_s = \frac{1}{100\pi}$ seconds.

Solution: The taps are given by:

k	a_k	b_k
1	0.9768	0.0666
2	0.9143	0.1616
3	0.8177	0.2803
4	0.6954	0.4156
5	0.5576	0.5576
6	0.4156	0.6954
7	0.2803	0.8177
8	0.1616	0.9143
9	0.0666	0.9768

2. *Derivative of Envelope Process.*

Hint: If $\{X(t)\}$ is a real-valued WSS process with $Y(t) = \frac{d}{dt}X(t)$, then

$$R_{XY}(\tau) = -\frac{d}{d\tau}R_X(\tau)\ , \text{ and } \ R_Y(\tau) = -\frac{d^2}{d\tau^2}R_X(\tau)\ .$$

Consider an isotropic Rayleigh fading process $E(t)$ with envelope $\beta(t)$ and phase $\phi(t)$.

(a) Show that the derivative of the envelope process is given by

$$\dot{\beta}(t) = \frac{d}{dt}\beta(t) = \dot{E}_I(t)\ \cos\phi(t) + \dot{E}_Q(t)\ \sin\phi(t)$$

where $\dot{E}_I(t)$ and $\dot{E}_Q(t)$ are the derivatives of the in-phase and quadrature processes, respectively.

(b) Show that $\dot{E}_I(t)$ and $\dot{E}_Q(t)$ are *mutually independent, zero mean, Gaussian* processes with

$$R_{\dot{E}_I}(\xi) = R_{\dot{E}_Q}(\xi) = \pi\,\nu_{\text{max}}^2 \int_{-\pi}^{\pi} \cos(2\pi\nu_{\text{max}}\xi\cos\theta)\ \cos^2\theta\ d\theta\ .$$

(c) For fixed t, show that $\dot{\beta}(t)$ is a Gaussian random variable with zero mean and variance $\pi^2\nu_{\text{max}}^2$. *Hint*: Condition on $\phi(t)$ first.

(d) For fixed t, show that $\beta(t)$ and $\dot{\beta}(t)$ are independent random variables, and write down their joint pdf.

Solution: We talked about this in class.

3. *Block Correlated Fading Model.* As we discussed in class (also see page 58 of the notes), the block (independent) fading model can be generalized to allow for both time-selectivity within the block ($TB_b > 1$) as well as correlation across blocks. For simplicity consider the frequency-flat case, and derive a reasonable model for correlation across blocks that reflects the large scale variations in the channel. You may assume that the median of the large scale variations is always normalized to 1 via power control.

Solution: See Y. Liang and V.V. Veeravalli. "Capacity of Noncoherent Time-Selective Block Fading Channels." IEEE Transactions on Information Theory, 50(12):3095-3110, December 2004, for one way to model the correlation across blocks.

4. *Useful results.* Show that

(a)

$$\int_0^\infty Q(\sqrt{x})\frac{e^{-x/\gamma}}{\gamma}dx = \frac{1}{2}\left(1-\sqrt{\frac{\gamma}{2+\gamma}}\right).$$

Solution:

$$\int_0^\infty \mathrm{Q}(\sqrt{x})\frac{e^{-\frac{x}{\gamma}}}{\gamma}dx = \int_0^\infty\int_{\sqrt{x}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\frac{e^{-\frac{x}{\gamma}}}{\gamma}dydx = \int_0^\infty\int_0^{y^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\frac{e^{-\frac{x}{\gamma}}}{\gamma}dxdy$$
$$= \int_0^\infty \frac{1}{\sqrt{2\pi}}(1-e^{-\frac{y^2}{\gamma}})e^{-\frac{y^2}{2}}dy$$
$$= \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy - \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{\gamma}}e^{-\frac{y^2}{2}}dy = \frac{1}{2}-\frac{1}{2}\sqrt{\frac{\gamma}{2+\gamma}}$$

(b) For $t>0$,

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta.$$

This is an alternative, and useful form, for the $Q(\cdot)$ function.

Solution:

$$\mathrm{Q}(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx = \int_{-\infty}^\infty\int_0^\infty \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy = \int_{\frac{t}{\cos\theta}}^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2\pi}e^{-\frac{r^2}{2}}rdrd\theta$$
$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-\frac{t^2}{2\cos^2\theta}}d\theta = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}e^{-\frac{t^2}{2\cos^2\theta}}d\theta = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}e^{-\frac{t^2}{2\sin^2\theta}}d\theta$$

5. *MPSK in Rayleigh fading.*

For MPSK signaling,

$$\mathrm{P_e}(\gamma_\mathrm{s}) = \frac{1}{\pi}\int_0^{(M-1)\pi/M}\exp\left[-\frac{\gamma_\mathrm{s}\sin^2(\pi/M)}{\sin^2\theta}\right]d\theta\ .$$

Using this expression how that the average symbol error probability $\overline{\mathrm{P}}_\mathrm{e}$ for MPSK signaling in Rayleigh fading is given in closed form by

$$\overline{\mathrm{P}}_\mathrm{e} = \left(1-\frac{1}{M}\right) - \frac{1}{\sqrt{1+a^2}}\left[\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{\cot\pi/M}{\sqrt{1+a^2}}\right)\right],$$

where $a^2 = \frac{1}{\overline{\gamma}_\mathrm{s}\sin^2\pi/M}$.

Hint: You may need to use the following integral

$$\int_{\theta_1}^{\theta_2}\frac{1}{\mathrm{cosec}^2\theta+a^2}d\theta = \frac{1}{a^2}\left[\frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{\cot\theta}{\sqrt{1+a^2}}\right)-\left(\frac{\pi}{2}-\theta\right)\right]_{\theta_1}^{\theta_2}\ \text{for } 0\le\theta_1\le\theta_2\le\pi/2.$$

Solution:

$$\overline{\mathrm{P}}_{\mathrm{e}} = \int_0^\infty \mathrm{P}_{\mathrm{e}}(x) p_{\gamma_{\mathrm{s}}}(x) dx = \frac{1}{\pi}\int_0^\infty \int_0^{\pi-\frac{\pi}{M}} e^{-\frac{x\sin^2\pi/M}{\sin^2\theta}} d\theta \, \frac{1}{\overline{\gamma}_{\mathrm{s}}} e^{-\frac{x}{\overline{\gamma}_{\mathrm{s}}}} dx$$

$$= \frac{1}{\pi}\int_0^{\pi-\frac{\pi}{M}} \frac{1}{\overline{\gamma}_{\mathrm{s}}}\int_0^\infty e^{-x\left(\frac{\sin^2\pi/M}{\sin^2\theta}+\frac{1}{\overline{\gamma}_{\mathrm{s}}}\right)} dx d\theta = \frac{1}{\pi\overline{\gamma}_{\mathrm{s}}}\int_0^{\pi-\frac{\pi}{M}} \frac{1}{\frac{\sin^2\pi/M}{\sin^2\theta}+\frac{1}{\overline{\gamma}_{\mathrm{s}}}} d\theta$$

$$= \frac{1}{\pi\overline{\gamma}_{\mathrm{s}}\sin^2\pi/M}\int_0^{\frac{(M-1)\pi}{M}} \frac{1}{\csc^2\theta + \frac{1}{\overline{\gamma}_{\mathrm{s}}\sin^2\pi/M}} d\theta = \frac{a^2}{\pi}\int_0^{\frac{(M-1)\pi}{M}} \frac{1}{\csc^2\theta + a^2} d\theta$$

$$= \frac{1}{\pi}\left[\frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{\cot(\pi-\pi/M)}{\sqrt{1+a^2}}\right) - \frac{\pi}{2\sqrt{1+a^2}} + \pi - \frac{\pi}{M}\right]$$

$$= \left(1-\frac{1}{M}\right) - \frac{1}{\sqrt{1+a^2}}\left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\cot\pi/M}{\sqrt{1+a^2}}\right)\right]$$

where $a^{-1} = \sqrt{\overline{\gamma}_{\mathrm{s}}}\sin\pi/M$.

6. *Non-coherent M-ary orthogonal signaling in Ricean fading*:

For Non-coherent M-ary orthogonal signaling,

$$\mathrm{P}_{\mathrm{e}}(\gamma_{\mathrm{s}}) = \sum_{n=1}^{M-1} (-1)^{n+1}\binom{M-1}{n}\frac{1}{n+1}\exp\left[-\frac{n\gamma_{\mathrm{s}}}{(n+1)}\right].$$

Using this expression, show that $\overline{\mathrm{P}}_{\mathrm{e}}$ under Ricean fading with Rice factor κ is given by

$$\overline{\mathrm{P}}_{\mathrm{e}} = \sum_{n=1}^{M-1} (-1)^{n+1}\binom{M-1}{n}\frac{\tilde{p}(n,\overline{\gamma}_{\mathrm{s}})}{n+1},$$

where

$$\tilde{p}(n,\overline{\gamma}_{\mathrm{s}}) = \frac{\kappa+1}{\kappa+1+\frac{n}{n+1}\overline{\gamma}_{\mathrm{s}}}\exp\left(-\frac{\frac{n}{n+1}\overline{\gamma}_{\mathrm{s}}\,\kappa}{\kappa+1+\frac{n}{n+1}\overline{\gamma}_{\mathrm{s}}}\right)$$

Solution:

$$\overline{\mathrm{P}}_{\mathrm{e}} = \int_0^\infty \mathrm{P}_{\mathrm{e}}(x) p_{\gamma_{\mathrm{s}}}(x) dx$$

For Ricean fading with Rice factor κ,

$$p_{\gamma_{\mathrm{s}}}(x) = \frac{\kappa+1}{\overline{\gamma}_{\mathrm{s}}} I_0\left(2\sqrt{\frac{\kappa(\kappa+1)x}{\overline{\gamma}_{\mathrm{s}}}}\right) e^{-\kappa-\frac{(\kappa+1)x}{\overline{\gamma}_{\mathrm{s}}}} \mathbb{1}_{\{x\geq 0\}}$$

Now,

$$\int_0^\infty e^{-\frac{nx}{n+1}} p_{\gamma_{\mathrm{s}}}(x) dx = \frac{\kappa+1}{\kappa+1+\frac{n}{n+1}\overline{\gamma}_{\mathrm{s}}} e^{-\frac{\frac{\kappa n k}{n+1}\bar{r}_b}{\kappa+1+\frac{n}{n+1}\overline{\gamma}_{\mathrm{s}}}} = \tilde{p}(n,\bar{r}_b)$$

This follows by using the fact that a Ricean PDF must integrate to 1. The result follows.

7. *Nakagami-m fading.* The first order statistics of a flat fading channel are sometimes approximated by a pdf from the Nakagami-m family:

$$p_\beta(x) = \frac{2m^m x^{2m-1}}{\Gamma(m)a^m} \exp\left(-\frac{mx^2}{a}\right) \mathbb{1}_{\{x>0\}}, \quad m > 0.5$$

where $a = \mathrm{E}[\beta^2] = 1$, and $\Gamma(\cdot)$ is the Gamma function which is defined by the integral $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, for $x > 0$. (Properties of $\Gamma(\cdot)$ include: $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(0.5) = \sqrt{\pi}$, $\Gamma(1) = 1$, and $\Gamma(n+1) = n!$, for positive integer n.)

Note: If m is positive integer, which means that $\Gamma(m) = (m-1)!$, the above p.d.f. is a central chi-squared distribution with $2m$ degrees of freedom, which is also the p.d.f. of the sum of m independent and identically distributed exponential random variables – sometimes called an Erlang distribution.

(a) Show that the pdf of $\gamma_\mathrm{b} = \beta^2\overline{\gamma}_\mathrm{b}$ is given by

$$p_{\gamma_\mathrm{b}}(x) = \left(\frac{m}{\overline{\gamma}_\mathrm{b}}\right)^m \frac{x^{m-1}}{\Gamma(m)} \exp\left(-\frac{mx}{\overline{\gamma}_\mathrm{b}}\right) \mathbb{1}_{\{x>0\}}.$$

Solution:

$$p_{\gamma_\mathrm{b}}(x) = \frac{1}{2\sqrt{x\overline{\gamma}_\mathrm{b}}} p_\beta(\sqrt{x/\overline{\gamma}_\mathrm{b}}) = \frac{1}{2\sqrt{x\overline{\gamma}_\mathrm{b}}} \frac{2m^m(\sqrt{x/\overline{\gamma}_\mathrm{b}})^{2m-1}}{\Gamma(m)} \exp\left(-\frac{mx}{\overline{\gamma}_\mathrm{b}}\right) \mathbb{1}_{\{x>0\}}$$

(b) If m is a positive integer, show that the c.d.f. of γ_b is given by:

$$F_{\gamma_\mathrm{b}}(x) = \int_0^x p_{\gamma_\mathrm{b}}(y)dy = 1 - \sum_{i=0}^{m-1} \left(\frac{m}{\overline{\gamma}_\mathrm{b}}\right)^i \frac{x^i}{i!} \exp\left(-\frac{mx}{\overline{\gamma}_\mathrm{b}}\right).$$

Solution: When m is a positive integer, $\Gamma(m) = (m-1)!$,

$$F_{\gamma_\mathrm{b}}(y) = \int_0^y \left(\frac{m}{\overline{\gamma}_\mathrm{b}}\right)^m \frac{x^{m-1}}{(m-1)!} \exp\left(-\frac{mx}{\overline{\gamma}_\mathrm{b}}\right) dx$$

Integrating by parts m times, we get the result.

(c) Now show that if m is a positive integer, $\overline{\mathrm{P}}_\mathrm{b}$ for BPSK signaling in slow, flat Nakagami-m fading is given by

$$\overline{\mathrm{P}}_\mathrm{b} = \frac{1}{2} - \frac{\tilde{\sigma}}{2} \sum_{i=0}^{m-1} \left(\frac{m\tilde{\sigma}^2}{4\overline{\gamma}_\mathrm{b}}\right)^i \frac{(2i)!}{(i!)^2},$$

where $\tilde{\sigma} = \sqrt{\overline{\gamma}_\mathrm{b}/(\overline{\gamma}_\mathrm{b} + m)}$.

Hint: You may want to use the fact that the even moments of a $\mathcal{N}(0,1)$ random variable X are given by:

$$\mathrm{E}[X^{2i}] = \frac{(2i)!}{i!\ 2^i}.$$

Solution:

$$
\begin{aligned}
\mathrm{P_b} &= \int_0^\infty \mathrm{Q}(\sqrt{2x}) p_{\gamma_\mathrm{b}}(x) dx = \int_0^\infty p_{\gamma_\mathrm{b}}(x) \left(\int_{\sqrt{2x}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \right) dx \\
&= \int_0^\infty \int_0^{\frac{y^2}{2}} p_{\gamma_\mathrm{b}}(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dx dy = \int_0^\infty F_{\gamma_\mathrm{b}} \left(\frac{y^2}{2} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&= \frac{1}{2} - \sum_{i=0}^{m-1} \left(\frac{m}{2\overline{\gamma}_\mathrm{b}} \right)^i \frac{1}{i!} \int_0^\infty y^{2i} \, \frac{e^{-\frac{y^2}{2}\left(\frac{m}{\overline{\gamma}}+1\right)}}{\sqrt{2\pi}} dy
\end{aligned}
$$

Now,

$$
\int_0^\infty y^{2i} \frac{e^{-\frac{y^2}{2}\left(\frac{m}{\bar{r}_b}+1\right)}}{\sqrt{2\pi}} dy = \tilde{\sigma} \int_0^\infty \tilde{\sigma}^{2i} u^{2i} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du = \frac{1}{2} \tilde{\sigma}^{2i+1} \frac{(2i)!}{(i!)^2}
$$

from which the final result follows.