ECE 559 **Fall 2007**

Handout # October 9, 2007

HOMEWORK ASSIGNMENT 3 (due October 19 (Friday))

1. *Optimality of maximal-ratio combining scheme for coherent detection with diversity*: Consider BPSK signaling on an L-th order diversity channel. Each channel introduces a fixed attenuation and phase shift so that the received signal at the output of the ℓ-th channel is:

$$y_\ell(t) = \pm \alpha_\ell e^{j\phi_\ell} \sqrt{\mathcal{E}}\, g_\ell(t) + w_\ell(t)$$

where the processes $w_\ell(t)$ are independent complex WGN processes with PSD N_0.

The receiver uses the decision statistic

$$y = \sum_{\ell=1}^{L} \beta_\ell \,\langle y_\ell(t), g_\ell(t) \rangle$$

where the $\{\beta_\ell\}$ are complex weighting factors to be determined. A decision in favor of $+1$ ("bit 1") is made if $y_I > 0$ and -1 ("bit 0") otherwise.

   (a) Determine the p.d.f. of y_I when $+1$ is transmitted.

   (b) Show that the probability of bit error $\mathrm{P_b}$ is given by:

$$\mathrm{P_b} = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}} \quad \frac{\sum_{\ell=1}^{L} \mathrm{Re}\{\beta_\ell \alpha_\ell e^{j\phi_\ell}\}}{\sqrt{\sum_{\ell=1}^{L} |\beta_\ell|^2}} \right).$$

   (c) Determine the values of $\{\beta_\ell\}$ that minimize $\mathrm{P_b}$.
   Hint: Use the Cauchy-Schwarz inequality.

2. *Equal gain combining*: Consider the same diversity channel model as in problem 1. Suppose the receiver is able to obtain perfect estimates of the phases $\{\phi_\ell\}$ but does not have estimates of $\{\alpha_\ell\}$. A useful test statistic is formed by equal gain combining that yields:

$$y = \sum_{\ell=1}^{L} e^{-j\phi_\ell} \,\langle y_\ell(t), g_\ell(t) \rangle$$

and a decision in favor of $+1$ ("bit 1") is made if $y_I > 0$ and -1 ("bit 0") otherwise.

   (a) Determine the p.d.f. of y_I when $+1$ is transmitted, for fixed values of $\{\alpha_\ell\}$.

   (b) Find the probability of bit error $\mathrm{P_b}$, for fixed values of $\{\alpha_\ell\}$.

Note: In class we analyzed the average bit error probability for maximum ratio combining under the assumption that the $\{\alpha_\ell\}$ are i.i.d. Ricean (or Rayleigh) random variables. It would be nice if we could extend this analysis to equal gain combining. Unfortunately, it is not possible to simplify the expression for $\overline{\mathrm{P}}_\mathrm{b}$ in this case, and we are left with a multi-dimensional integral over the joint pdf of the $\{\alpha_\ell\}$ (why?). Of course, it is possible (but not necessary for this homework!) to obtain performance results by computing the multidimensional integral either directly or via Monte-Carlo techniques.

3. *Diversity combining with Non-coherent detection.* Consider a binary FSK signal with Rayleigh fading. Let the received signal at the output of the ℓ-th channel be

$$y_\ell(t) = \alpha_\ell e^{j\phi_\ell}\sqrt{\mathcal{E}}\, g_{m,\ell}(t) + w_\ell(t), \quad \ell = 1, 2, \ldots, L, \quad m = 0, 1,$$

where $g_{0,\ell}(t)$ and $g_{1,\ell}(t)$ are orthogonal

The statistics $y_{m,\ell} = \langle y_\ell(t), g_{m,\ell}(t)\rangle$ for $\ell = 1, 2, \ldots, L, \quad m = 0, 1$ are sufficient.

We do not assume knowledge of $\{\alpha_\ell\}$ or $\{\phi_\ell\}$ at the receiver. Now, if we model the $\{\phi_\ell\}$ as i.i.d. Uniform$[0, 2\pi]$, then it is possible to show that MPE receiver (for uniform priors on m) can be shown to be constructed as follows. First, we compute the statistics

$$V_m = \frac{1}{N_0}\sum_{\ell=1}^{L} |y_{m,\ell}|^2 \quad m = 0, 1,$$

and then decide '0' if $V_0 > V_1$, and '1' otherwise. (The $1/N_0$ factor is just for normalization.)

(a) Assuming that the fading is independent (Rayleigh) across the channels, show that, conditioned on '0' being sent, the pdf's of V_0 and V_1 are given by

$$p_{V_0}(x) = \frac{x^{L-1}}{(L-1)!(1+\overline{\gamma})^L}\exp\left(-\frac{x}{\overline{\gamma}+1}\right)\mathbb{1}_{\{x\geq 0\}}, \text{ and } p_{V_1}(x) = \frac{x^{L-1}}{(L-1)!}e^{-x}\mathbb{1}_{\{x\geq 0\}},$$

where $\overline{\gamma} = \mathcal{E}/N_0$.

(b) It is easy to see that

$$\overline{\mathrm{P}}_\mathrm{b} = \mathrm{P}\left(\{V_1 > V_0\} \mid \{\text{'0' sent}\}\right) = 1 - \int_0^\infty F_{V_1}(x)p_{V_0}(x)dx$$

where $F_{V_1}(\cdot)$ is the cdf of V_1. Based on problem 7 (b) of HW#2, you should be able to write down an equation for $F_{V_1}(x)$.

Now show that

$$\overline{\mathrm{P}}_\mathrm{b} = \left(\frac{1}{2+\overline{\gamma}}\right)^L \sum_{\ell=0}^{L-1}\binom{L-1+\ell}{\ell}\left(\frac{1+\overline{\gamma}}{2+\overline{\gamma}}\right)^\ell.$$

(c) Using the fact that $\overline{\gamma} = \overline{\gamma}_\mathrm{b}/L$, plot $\overline{\mathrm{P}}_\mathrm{b}$ vs. $\overline{\gamma}_\mathrm{b}$ for $\overline{\gamma}_\mathrm{b}$ ranging from 5 to 40 dB, and for $L = 2, 3, 4$.

4. *BPSK with perfect interleaving and hard decision decoding on a Rayleigh fading channel.*

(a) Show that

$$\overline{\mathrm{P}}_\mathrm{b} \leq \overline{\mathrm{P}}_\mathrm{c\,e} \leq \sum_{q=t+1}^{n}\binom{n}{q}\left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}_\mathrm{c}}{1+\overline{\gamma}_\mathrm{c}}}\right)^q\left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\overline{\gamma}_\mathrm{c}}{1+\overline{\gamma}_\mathrm{c}}}\right)^{n-q},$$

where $t = \lfloor\frac{d_\mathrm{min}-1}{2}\rfloor$ (d_min is the minimum distance of the code).

(b) Plot the bound on $\overline{\mathrm{P}}_\mathrm{b}$ as a function of $\overline{\gamma}_\mathrm{b}$ (ranging from 0 to 30 dB) for a (7,4) Hamming code. (Note that $\overline{\gamma}_\mathrm{b} = 7\overline{\gamma}_\mathrm{c}/4$ in this case.)

5. *BPSK with perfect interleaving and soft decision decoding on a Rayleigh fading channel.*

(a) Show that

$$\overline{\mathrm{P}}_{\mathrm{b}} \leq \overline{\mathrm{P}}_{\mathrm{c\,e}} \leq \sum_{j=2}^{2^k} \left(\frac{1-\tilde{\sigma}}{2}\right)^{\omega_j} \sum_{q=0}^{\omega_j-1} \binom{\omega_j - 1 + q}{q} \left(\frac{1+\tilde{\sigma}}{2}\right)^{q},$$

where $\tilde{\sigma} = \sqrt{\frac{\overline{\gamma}_{\mathrm{c}}}{\overline{\gamma}_{\mathrm{c}}+1}}$ and ω_j is the Hamming weight of $\mathbf{c}_j$.

(b) Plot the bound on $\overline{\mathrm{P}}_{\mathrm{b}}$ as a function of $\overline{\gamma}_{\mathrm{b}}$(ranging from 0 to 30 dB) for a (7,4) Hamming code using the weight distribution given to you in class.