SOLUTIONS TO HW # 3

1. (i) $y = \sum_{l=1}^{L} \beta_l \langle y_l(t), g_l(t) \rangle = \pm \sum_{l=1}^{L} \beta_l \alpha_l e^{j\phi_l} \sqrt{\mathcal{E}} + \sum_{l=1}^{L} w_l$

where $w_l \sim \mathcal{CN}(0, N_0 |\beta_l|^2)$ and $\{w_l\}_{l=1}^{L}$ are independent

Thus

$$y_I \sim \mathcal{N}\left(\sum_{l=1}^{L} \alpha_l \sqrt{\mathcal{E}}\, \mathrm{Re}[\beta_l e^{j\phi_l}],\ \frac{N_0}{2} \sum_{l=1}^{L} |\beta_l|^2 \right)$$

when '+1' is transmitted

(ii) $P_b = P\left(\{ y_I < 0 \} \mid \{+1 \text{ transmitted}\} \right)$

$$= Q\left(\frac{\sum_{l=1}^{L} \sqrt{\mathcal{E}}\ \mathrm{Re}[\beta_l \alpha_l e^{j\phi_l}]}{\sqrt{\sum_{l=1}^{L} \frac{N_0}{2} |\beta_l|^2}} \right) = \text{answer given}$$

(iii) Let $\underline{\beta} = [\beta_1\ \beta_2\ \cdots\ \beta_L]^T$ and $\underline{e} = [\alpha_1 e^{-j\phi_1}\ \cdots\ \alpha_L e^{-j\phi_L}]^T$

$$\langle \underline{\beta}, \underline{e} \rangle = \underline{e}^{+} \underline{\beta},\quad \langle \underline{e}, \underline{e} \rangle = \sum_{l=1}^{L} \alpha_l^2,\quad \langle \underline{\beta}, \underline{\beta} \rangle = \sum_{l=1}^{L} |\beta_l|^2$$

By C-S inequality $\quad |\langle \underline{\beta}, \underline{e} \rangle| \le \sqrt{\langle \underline{\beta}, \underline{\beta} \rangle} \sqrt{\langle \underline{e}, \underline{e} \rangle}$

with equality iff $\underline{\beta} = c\, \underline{e}$ for some scalar $c \neq 0$.

Thus

$$\frac{\sum_{l=1}^{L} \mathrm{Re}[\beta_l \alpha_l e^{j\phi_l}]}{\sqrt{\sum_{l=1}^{L} |\beta_l|^2}} = \frac{\mathrm{Re}[\langle \underline{\beta}, \underline{e} \rangle]}{\sqrt{\langle \underline{\beta}, \underline{\beta} \rangle}} \le \frac{|\langle \underline{\beta}, \underline{e} \rangle|}{\sqrt{\langle \underline{\beta}, \underline{\beta} \rangle}} \le \sqrt{\langle \underline{e}, \underline{e} \rangle}$$

with equality in both inequalities iff c is real and $\neq 0$

Without loss of generality, we can pick $c = 1$ and hence we get that the β_l that maximize the argument of the $Q(\cdot)$ function and hence minimize P_b are:

$$\beta_l = \alpha_l e^{-j\phi_l}, \quad l = 1, \ldots, L$$

2. (i) $y = \pm \sum_{\ell=1}^{L} \sqrt{\mathcal{E}}\, \alpha_\ell + \sum_{\ell=1}^{L} w_\ell$

where $w_\ell \sim \mathcal{CN}(0, N_0)$ and $\{w_\ell\}_{\ell=1}^{L}$ are i.i.d.

Thus, when $+1$ is transmitted,

$$y_I \sim \mathcal{N}\left(\sum_{\ell=1}^{L} \sqrt{\mathcal{E}}\, \alpha_\ell \,,\; L \frac{N_0}{2} \right)$$

(ii) For fixed values of $\{\alpha_\ell\}$,

$$P_b = P\left(\{ y_I < 0\} \mid \{+1 \text{ trans}\}\right)$$

$$= Q\left(\frac{\sum_\ell \sqrt{\mathcal{E}}\, \alpha_\ell}{\sqrt{\frac{L N_0}{2}}} \right) = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0 L}} \sum_{\ell=1}^{L} \alpha_\ell \right)$$

3. If '0' is sent,

$$y_{0,\ell} = \alpha_\ell e^{j\phi_\ell} \sqrt{\mathcal{E}} + w_{0,\ell}$$

and $\quad y_{1,\ell} = w_{1,\ell}$

Now, by the orthogonality of $g_{0,\ell}(t)$ and $g_{1,\ell}(t)$, and the independence across channels, it is clear that

$y_{0,1}, \ldots, y_{0,L}, y_{1,1}, \ldots, y_{1,L}$, are independent RV's

Furthermore, $y_{0,\ell}$ is a PCG random variable since $\alpha_\ell e^{j\phi_\ell}$ and $w_{0,\ell}$ are independent PCG random variables

$$E[y_{0,\ell}] = E[\alpha_\ell e^{j\phi_\ell}] \sqrt{\mathcal{E}} + E[w_{0,\ell}] \underset{\uparrow}{=} 0$$

Rayleigh fading

$$E[\,|y_{0,\ell}|^2] = \mathcal{E}\; E[\alpha_\ell^2] + N_0 = \mathcal{E} + N_0$$

Thus V_0 is the sum of L i.i.d. exponential random variables, each with mean $\frac{1}{N_0}(\mathcal{E} + N_0) = (\bar{\gamma} + 1)$

Hence (see prob 6 of HW#4),

$$p_{V_0}(x) = \frac{x^{L-1}}{(L-1)!\,(1+\bar{\gamma})^L} \exp\left(\frac{-x}{\bar{\gamma}+1} \right) \mathbb{1}_{\{x \geq 0\}}$$

To get $p_{V_1}(x)$ we simply set $\bar{\gamma}$ to zero in $p_{V_0}(x)$ to get:

$$p_{V_1}(x) = \frac{x^{L-1}}{(L-1)!} e^{-x} \; \mathbb{1}\{x \geq 0\}$$

(ii) $$\bar{P}_b = P(\{V_1 > V_0\} \mid \{\text{'0' sent}\}) = 1 - \int_0^\infty F_{V_1}(x)\, p_{V_0}(x)\, dx$$

Based on 6(ii) of HW # 4,

$$F_{V_1}(x) = 1 - \sum_{i=0}^{L-1} \frac{x^i}{i!} e^{-x}$$

Thus,

$$\bar{P}_b = \sum_{i=0}^{L-1} \int_0^\infty \frac{x^i}{i!} e^{-x} \frac{x^{L-1}}{(L-1)!} \frac{e^{-\frac{x}{\bar{\gamma}+1}}}{(1+\bar{\gamma})^L} dx$$

$$= \frac{1}{(1+\bar{\gamma})^L} \sum_{i=0}^{L-1} \frac{(L-1+i)!}{i!\,(L-1)!} \int_0^\infty \frac{x^{L-1+i}}{(L-1+i)!} e^{-\frac{x(\bar{\gamma}+2)}{\bar{\gamma}+1}} dx$$

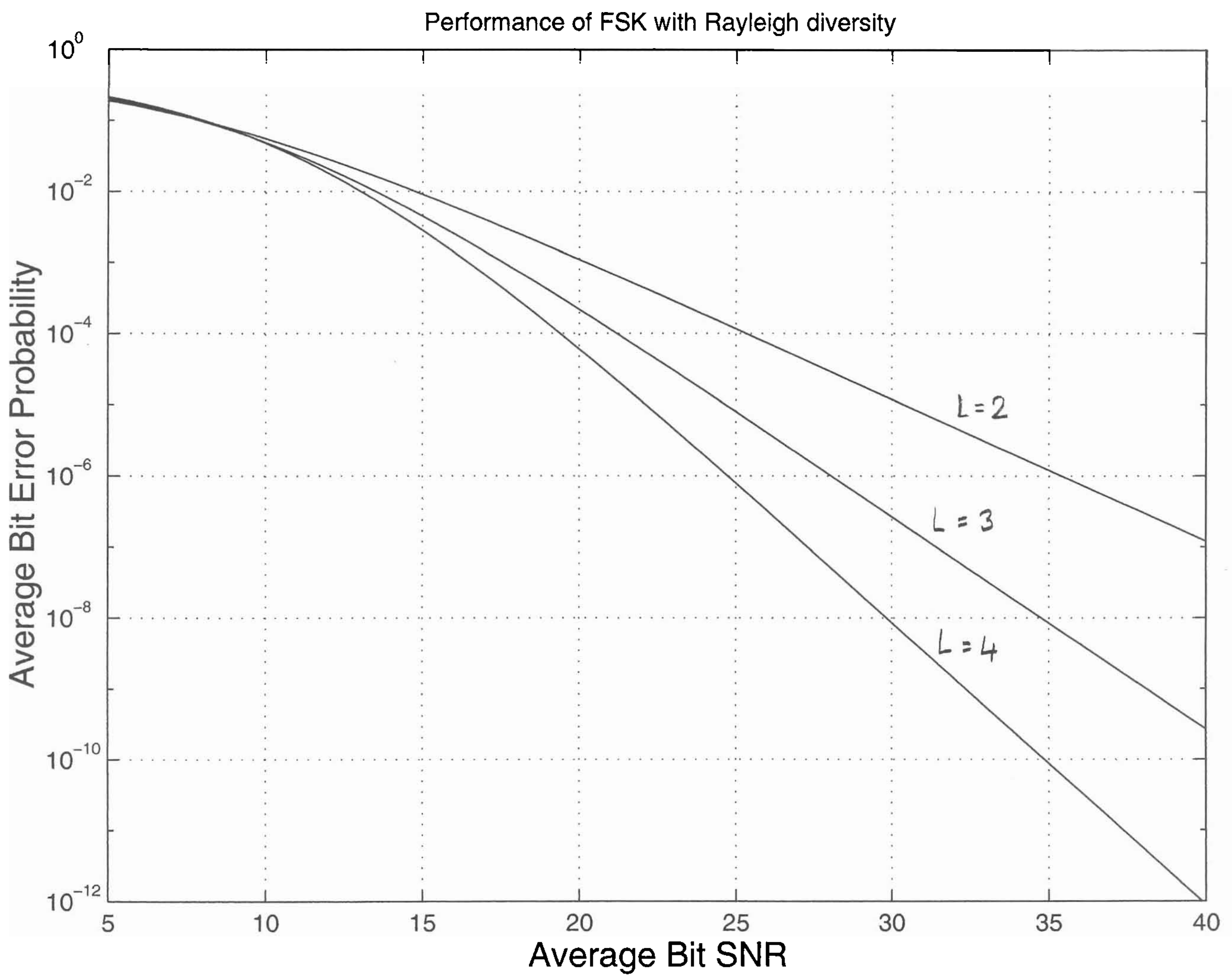
$$= \frac{1}{(1+\bar{\gamma})^L} \left(\frac{\bar{\gamma}+1}{\bar{\gamma}+2}\right)^L \sum_{i=0}^{L-1} \binom{L-1+i}{i} \left(\frac{\bar{\gamma}+1}{\bar{\gamma}+2}\right)^i \int_0^\infty \frac{e^{-u} u^{L+i-1}}{(L+i-1)!} du$$

$$= \left(\frac{1}{\bar{\gamma}+2}\right)^L \sum_{i=0}^{L-1} \binom{L-1+i}{i} \left(\frac{\bar{\gamma}+1}{\bar{\gamma}+2}\right)^i$$

(iii) See attached figure

4 & 5. (i) straightforward based on class notes

(ii) see Fig. 19.1 of notes


Performance of FSK with Rayleigh diversity
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
Average Bit Error Probability
5
10
15
20
25
30
35
40
Average Bit SNR
L = 2
L = 3
L = 4