## HOMEWORK ASSIGNMENT 4 (due November 7)

1. *Performance analysis of MF detection for asynchronous band-limited CDMA*. The received signal on the reverse link in the BPSK/CDMA system can be written as

$$y(t) = \sum_{k=1}^{K} \sum_{i=-\infty}^{\infty} \sqrt{\mathcal{E}_{\mathrm{b},k}}\, z_k^{(\lfloor i/N \rfloor)}\, c_{k,i}\, g_{T_c}(t - iT_c - \tau_k)\, e^{j\phi_k} + w(t)$$

where $w(t)$ is CCG($N_0$), $N$ is the processing gain, and $g_{T_c}(t)$ is a unit energy chip waveform with the auto-correlation $R_g(d) = \int_{-\infty}^{\infty} g_{T_c}(t) g_{T_c}(t - dT_c) dt$. The pulse shape is assumed to have zero inter-chip interference (ICI) under perfect sampling, so that $R_g(m-n) = \delta[m-n]$. Note that the chip sequences $\{c_{k,i}\}_{i=-\infty}^{\infty}$ are not divided into blocks of length $N$ – they are assumed to be long (random) binary sequences taking values in $\{-\frac{1}{\sqrt{N}}, +\frac{1}{\sqrt{N}}\}$.

The matched filter statistic for detecting the bit $z_1^{(0)}$ of user 1 is given by

$$X_{\text{MF}} = \text{Re}\left\{ \left\langle y(t)\ ,\ e^{j\phi_1} \left[ \sum_{m=0}^{N-1} c_{1,m}\, g_{T_c}(t - mT_c - \tau_1) \right] \right\rangle \right\}$$

and $\hat{z}_1^{(0)} = \text{sgn}(X_{\text{MF}})$.

(i) Assuming w.l.o.g. that $\tau_1 = 0$, show that

$$X_{\text{MF}} = A_1 z_1^{(0)} + \sum_{k=2}^{K} \sum_{m=0}^{N-1} c_{1,m}\, y_k[m] + w_1$$

where $w_1 \sim \mathcal{N}(0, N_0/2)$, $A_k = \sqrt{\mathcal{E}_{\mathrm{b},k}}$ and

$$y_k[m] = \sum_{i=-\infty}^{\infty} A_k\, z_k^{(\lfloor i/N \rfloor)}\, c_{k,i}\, R_g(i + d_k - m)\, \cos(\phi_k - \phi_1), \quad \text{with } d_k = \tau_k/T_c.$$

(ii) *Moments.* Define $X_k = \sum_{m=0}^{N-1} c_{1,m} y_k[m]$. Assume that the bits and chip sequences are all *i.i.d.* equally likely Bernouilli random variables, and the phases are *i.i.d.* Unif$[0, 2\pi]$. Show that $\text{E}(X_k) = 0$ and that

$$\text{E}(X_k^2 | d_k) = \frac{A_k^2}{2N^2} \sum_{m=0}^{N-1} \sum_{i=-\infty}^{\infty} R_g^2(i - m + d_k). \tag{1}$$

Clearly, the integral part of the $d_k$ does not make a difference to the summation. Hence, model $\{d_k\}$ as *i.i.d.* Unif$[0, 1]$ random variables, and remove the conditioning on $d_k$ to arrive at

$$\text{E}(X_k^2) = \frac{A_k^2}{2N} \int_{-\infty}^{\infty} R_g^2(u)\, \mathrm{d}u = \frac{A_k^2 \sigma_g^2}{2N}$$

where $\sigma_g^2 = \int_{-\infty}^{\infty} R_g^2(u) \mathrm{d}u$ is a function of the chip pulse shape.

 

(iii) *Error Probability.* Assuming that $X_{\mathrm{MF}}$ is Gaussian, show that the bit error probability is given by

$$\mathrm{P_b} = Q(A_1/\sigma)$$

where $\sigma^2 = \sum_{k=2}^{K} \frac{A_k^2 \sigma_g^2}{2N} + \frac{N_0}{2}$. Consequently, for large $K$ and equal powers, $\mathrm{P_b} \to Q\left(\sqrt{\frac{2N}{(K-1)\sigma_g^2}}\right)$.

(iv) *Rectangular vs. sinc pulse.* What are the values of $\sigma_g^2$ for the rectangular pulse, $g_{T_c}(t) = \frac{1}{\sqrt{T_c}}\mathrm{rect}(t/T_c - 0.5)$ and the sinc pulse $g_{T_c}(t) = \frac{1}{\sqrt{T_c}}\mathrm{sinc}(t/T_c - 0.5)$. What does this imply about the effect of the chip waveform on the user capacity for a given error probability ?

(v) *Synchronous vs. Asynchronous.* When the users are (chip) synchronous, $d_k = 0\ \forall k$ in (1). Then show that the asymptotic error probability for large $K$ is $\mathrm{P_b} = Q\left(\sqrt{\frac{2N}{(K-1)}}\right)$. How does this compare with the error probability in the asynchronous band-limited case ?

2. Carefully derive the SIR and BER expressions for the single user MF detector for synchronous users with random spreading sequences given in the lecture notes on CDMA. Show all your steps clearly.

3. *Coding-Spreading tradeoff.* On the reverse link of a CDMA system, it may be harder to track phase variations (since there is no pilot reference), and orthogonal modulation techniques may be employed. Since orthogonal modulation leads to bandwidth expansion, the spreading factor needs to be decreased to keep the overall expansion factor($\Omega$) the same.

   For illustration, consider a $K$-user (equal power) synchronous system with $M$-ary orthogonal modulation followed by spreading with a processing gain $N$. Assume that the orthogonal modulation maps $\nu$ bits to $M = 2^\nu$ Walsh sequences (as done, for example, in the Qualcomm IS-95 system). Each Walsh chip is then spread by a *random* spreading sequence of length $N$ chips. At the receiver, the matched-filter soft output for each Walsh chip is fed to a non-coherent demodulator.

   (i) Using SIR expression derived in the previous problem show that the effective bit signal-to-interference ratio at the input to the orthogonal demodulator is given by

$$\gamma = \frac{\frac{\mathcal{E}_\mathrm{b}}{N_0}}{\frac{(K-1)\nu}{NM}\frac{\mathcal{E}_\mathrm{b}}{N_0} + 1}\,. \tag{2}$$

   (ii) Argue that $MN/\nu$ is the total bandwidth expansion factor $\Omega$. Thus, for fixed $\Omega$, $\gamma$ depends only on $\mathcal{E}_\mathrm{b}/N_0$ and the number of users $K$.

   Now express the bit error probability $\mathrm{P_b}$ at the output of orthogonal demodulator as a function of $\gamma$ (you may use the expression for $\mathrm{P_e}$ given in Problem 6 of HW#2 for this purpose). Furthermore, express $\mathrm{P_b}$ as a function of $\mathcal{E}_\mathrm{b}/N_0$ using (2).

   Plot $\mathrm{P_b}$ versus $\mathcal{E}_\mathrm{b}/N_0$ for $K = 5, 15$ and $M = 2, 8, 32$, fixing $\Omega$ at 32. What can we infer about the best choice for $M$? Also plot the curve $\mathrm{P_b} = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$ for single user BPSK.

   (iii) How would the choice of $M$ be affected if we used orthogonal sequences instead (as might happen on the forward link)? How would it be affected if a multi-user detector is used instead of the matched filter? Discuss.

4. Show that the recursion for the symbol estimates for the multistage approximation to the ML detector is given by:

$$\hat{z}_k(m+1) = \arg\max_{z_k}\ \mathrm{Re}\left[y_k A_k z_k^\star - \sum_{\ell \neq k} \rho_{\ell,k}\, A_\ell\, \hat{z}_\ell(m)\, A_k\, z_k^\star\right] - \frac{A_k^2\,|z_k|^2}{2}\,.$$

If the signaling is binary, show that the above expression reduces to:

$$\hat{z}_k(m+1) = \text{sgn}\left(\text{Re}\left[y_k - \sum_{\ell \neq k} \rho_{\ell,k}\, A_\ell\, \hat{z}_\ell(m)\right]\right) .$$

5. *Optimality of the Decorrelator.* Consider the MPE (ML) multiuser detector for a synchronous user system with binary symbols. We can write the ML solution as:

$$\hat{\mathbf{z}}_{\text{ML}} = \arg \max_{\mathbf{z} \in \{-1,+1\}^K} \left[2\mathbf{y}_I^\top \mathbf{A}\,\mathbf{z} - \mathbf{z}^\top \mathbf{A}\,\mathbf{R}\,\mathbf{A}\,\mathbf{z}\right] .$$

where $\mathbf{A} = \text{diag}(A_1, A_2, \ldots, A_K)$ with $A_k > 0$, and $\mathbf{y}_I = \text{Re}[\mathbf{y}]$.

Now suppose the $A_k$'s are unknown. Then we may pose the ML solution over both the $A_k$'s and the $z_k$'s to get:

$$\hat{\mathbf{z}}_{\text{opt}} = \arg \max_{\mathbf{z} \in \{-1,+1\}^K} \left\{\max_{\mathbf{A}: A_k > 0 \forall k} \left[2\mathbf{y}_I^\top \mathbf{A}\,\mathbf{z} - \mathbf{z}^\top \mathbf{A}\,\mathbf{R}\,\mathbf{A}\,\mathbf{z}\right]\right\}$$

Assuming that $\mathbf{R}^{-1}$ exists, show that the solution to the above optimization yields the decorrelating detector, i.e., that

$$\hat{\mathbf{z}}_{\text{opt}} = \text{sgn}\left[\mathbf{R}^{-1}\mathbf{y}_I\right] .$$