1 Error Control Coding for Fading Channels

Elements of Error Control Coding

- A *code* is a mapping that takes a sequence of information symbols and produces a (larger) sequence of code symbols so as to be able to detect/correct errors in the transmission of the symbols.

- The simplest class of codes is the class of *binary linear* block codes. Here each vector of k information bits $\mathbf{x}_i = [x_{i,1} \dots x_{i,k}]$ is mapped to vector of n code bits $\mathbf{c}_i = [c_{i,1} \dots c_{i,n}]$, with $n > k$. The rate R of the code is defined to be the ratio k/n.

- A binary linear block code can be defined in terms of a $k \times n$ generator matrix $\mathbf{G}$ with binary entries such that the code vector $\mathbf{c}_i$ corresponding to an information vector $\mathbf{x}_i$ is given by:

$$\mathbf{c}_i = \mathbf{x}_i \, \mathbf{G} \tag{1}$$

  (The multiplication and addition are the standard binary or GF(2) operations.)

- *Example:* $(7,4)$ Hamming Code

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & | & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & | & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{x}_i \, \mathbf{G} = \mathbf{c}_i \tag{2}$$

  Note that the codewords of this code are in *systematic* form with 4 information bits followed by 3 *parity* bits, i.e.,

$$\mathbf{c}_i = [x_{i,1} \; x_{i,2} \; x_{i,3} \; x_{i,4} \; c_{i,5} \; c_{i,6} \; c_{i,7}] \tag{3}$$

  with $c_{i,5} = x_{i,1} + x_{i,2} + x_{i,3}$, $c_{i,6} = x_{i,2} + x_{i,3} + x_{i,4}$, and $c_{i,7} = x_{i,1} + x_{i,2} + x_{i,4}$. It is easy to write down the 16 codewords of the (7,4) Hamming code. It is also easy to see that the minimum (Hamming) distance between the codewords, d_{min}, equals 3.

- *General Result.* If $d_{\text{min}} = 2t + 1$, then the code can correct t errors.

- *Example:* Repitition Codes. A rate-$\frac{1}{n}$ repitition code is defined by codebook:

$$0 \mapsto [0\,0 \dots 0] \text{ , and} 1 \mapsto [1\,1 \dots 1] \tag{4}$$

  The minimum distance of this code is n and hence it can correct $\lfloor \frac{n-1}{2} \rfloor$ errors. The optimum decoder for this code is simply a majority logic decoder. A rate-$\frac{1}{2}$ repitition code can detect one error, but cannot correct any errors. A rate-$\frac{1}{3}$ repitition code can correct one error.

Coding Gain

- The *coding gain* of a code is the gain in SNR, at a given error probability, that is achieved by using a code before modulation.

◦ The coding gain of a code is a function of: (i) the type of decoding , (ii) the error probability considered, (iii) the modulation scheme used, and (iv) the channel. We now compute the coding gain for BPSK signaling in AWGN for some simple codes. Before we proceed, we introduce the following notation:

$$\begin{aligned} \gamma_{\text{c}} &= \text{SNR per code bit} \\ \gamma_{\text{b}} &= \text{SNR per information bit} = \frac{\gamma_{\text{c}}}{R} \end{aligned}$$

◦ *Example:* Rate-$\frac{1}{2}$ repitition code, BPSK in AWGN

$$\mathsf{P}\{\text{code bit in error}\} = Q(\sqrt{2\gamma_{\text{c}}}\,) = Q(\sqrt{\gamma_{\text{b}}}\,) \tag{5}$$

For an AWGN channel, bit errors are independent across the codeword. It is easy to see that with majority logic decoding

$$\text{P}_{\text{ce}} = \mathsf{P}\{\text{decoding error}\} = [Q(\sqrt{\gamma_{\text{b}}}\,)]^2 + \frac{1}{2}\cdot 2\cdot Q(\sqrt{\gamma_{\text{b}}}\,)\,[1 - Q(\sqrt{\gamma_{\text{b}}}\,)] = Q(\sqrt{\gamma_{\text{b}}}\,)\,. \tag{6}$$

Thus

$$\text{P}_{\text{b}}(\text{with coding}) = Q(\sqrt{\gamma_{\text{b}}}\,) > Q(\sqrt{2\gamma_{\text{b}}}\,) = \text{P}_{\text{b}}(\text{without coding})\,. \tag{7}$$

The rate-$\frac{1}{2}$ repitition code results in a 3 dB coding loss for BPSK in AWGN at all error probabilities!

◦ *Example:* Rate-$\frac{1}{3}$ repitition code, BPSK in AWGN

$$\mathsf{P}\{\text{code bit in error}\} = Q(\sqrt{2\gamma_{\text{c}}}\,) = Q(\sqrt{2\gamma_{\text{b}}/3}\,) = p \;\; (\text{say})\,. \tag{8}$$

Then it is again easy to show that with majority logic decoding

$$\text{P}_{\text{b}}(\text{with coding}) = \text{P}_{\text{ce}} = p^3 + 3p^2(1-p) = 3p^2 - 2p^3 = 3\left[Q\left(\sqrt{\frac{2\gamma_{\text{b}}}{3}}\right)\right]^2 - 2\left[Q\left(\sqrt{\frac{2\gamma_{\text{b}}}{3}}\right)\right]^3 . \tag{9}$$

Furthermore one can show that the above expression for P_{ce} is always larger than P_{b} without coding (which equals $Q(\sqrt{2\gamma_{\text{b}}}\,)$). Hence this code also has a coding loss (negative coding gain) for BPSK in AWGN.

◦ *Example:* Rate-$\frac{1}{n}$ repitition code, BPSK in AWGN

$$\mathsf{P}\{\text{code bit in error}\} = Q(\sqrt{2\gamma_{\text{c}}}\,) = Q(\sqrt{2\gamma_{\text{b}}/n}\,) = p \;\; (\text{say})\,. \tag{10}$$

Then we can generalize the previous two examples to get:

$$\text{P}_{\text{b}}(\text{with coding}) = \text{P}_{\text{ce}} = \begin{cases} \sum_{q=\frac{n+1}{2}}^{n} \binom{n}{q}\, p^q\, (1-p)^{n-q} & \text{if } n \text{ is odd} \\ \frac{1}{2}\binom{n}{\frac{n}{2}}\, p^{\frac{n}{2}}\, (1-p)^{\frac{n}{2}} + \sum_{q=\frac{n}{2}+1}^{n} \binom{n}{q}\, p^q\, (1-p)^{n-q} & \text{if } n \text{ is even} \end{cases} \tag{11}$$

It is possible to show that the Rate-$\frac{1}{n}$ repitition code (with hard decision decoding) results in a coding loss for all n. (One way to show this is to see that even with optimum (soft-decision) decoding, the coding gain for a Rate-$\frac{1}{n}$ repitition code is 0.)

◦ *Example:* $(7,4)$-Hamming code, BPSK in AWGN

$$\mathsf{P}\{\text{code bit in error}\} = Q(\sqrt{2\gamma_\text{c}}\,) = Q(\sqrt{8\gamma_\text{b}/7}\,) = p \ \ (\text{say})\ . \tag{12}$$

Since the code can correct one code bit error (and will always have decoding error with more than one code bit error), we have

$$\text{P}_\text{ce} = \mathsf{P}\{2 \text{ or more code bits in error}\} = \sum_{q=2}^{7} \binom{7}{q} \, p^q \, (1-p)^{7-q}\ . \tag{13}$$

In general, it is difficult to find an exact relationship between the probability of information bit error P_b and the probability of codeword error P_ce. However, it is easy to see that $\text{P}_\text{b} \le \text{P}_\text{ce}$ always (see problem 4 of HW#5). Thus the above expression for P_ce serves as an upper bound for the P_b. Based on this bound, we can show (see Fig. 1) that for small enough P_b we obtain a positive coding gain from this code. Of course, this coding gain comes with a reduction in information rate (or bandwidth expansion).

◦ *Example:* Rate R, t-error correcting code, BPSK in AWGN

$$\mathsf{P}\{\text{code bit in error}\} = Q(\sqrt{2\gamma_\text{c}}\,) = Q(\sqrt{2R\gamma_\text{b}}\,) = p \ \ (\text{say})\ . \tag{14}$$

Now we can only bound P_ce since the code may not be "perfect" like the $(7,4)$-Hamming code. Thus

$$\text{P}_\text{b} \le \text{P}_\text{ce} \le \mathsf{P}\{t+1 \text{ or more code bits in error}\} = \sum_{q=t+1}^{n} \binom{n}{q} \, p^q \, (1-p)^{n-q}\ . \tag{15}$$

Soft decision decoding (for BPSK in AWGN)

$$\mathbf{x}_i = [x_{i,1} \dots x_{i,k}] \xrightarrow{\mathcal{C}} [c_{i,1} \dots c_{i,n}] = \mathbf{c}_i \tag{16}$$

◦ The code bits are sent using BPSK. If $c_{i,\ell} = 0$, -1 is sent; if $c_{i,\ell} = 1$, $+1$ is sent $\Rightarrow$ $(2c_{i,\ell} - 1)$ is sent. Thus, the received signal corresponding to the codeword $\mathbf{c}_i$ in AWGN is given by

$$y(t) = \sum_{\ell=1}^{n} (2c_{i,\ell} - 1)\ \sqrt{\mathcal{E}_\text{c}}\ g(t - \ell T_c) + w(t) \tag{17}$$

where T_c is the code bit period, and $g(\cdot)$ is a unit energy pulse shaping function that satisfies the zero-ISI condition w.r.t. T_c.

◦ The task of the decoder is to classify the received signal $y(t)$ into one of 2^k classes corresponding to the 2^k possible codewords. This is a 2^k-ary detection problem, for which the sufficient statistics are given by projecting $y(t)$ onto $g(t-\ell T_c)$, $\ell = 1, 2, \dots, n$. Alternatively, we could filter $y(t)$ with $g(T_c - t)$ and sample the output at rate $1/T_c$. The output of the matched filter for the ℓ-th code bit interval is given by:

$$y_\ell = (2c_{i,\ell} - 1)\sqrt{\mathcal{E}_\text{c}} + w_\ell \tag{18}$$

where $\{w_\ell\}$ are i.i.d. $\mathcal{CN}(0, N_0)$.

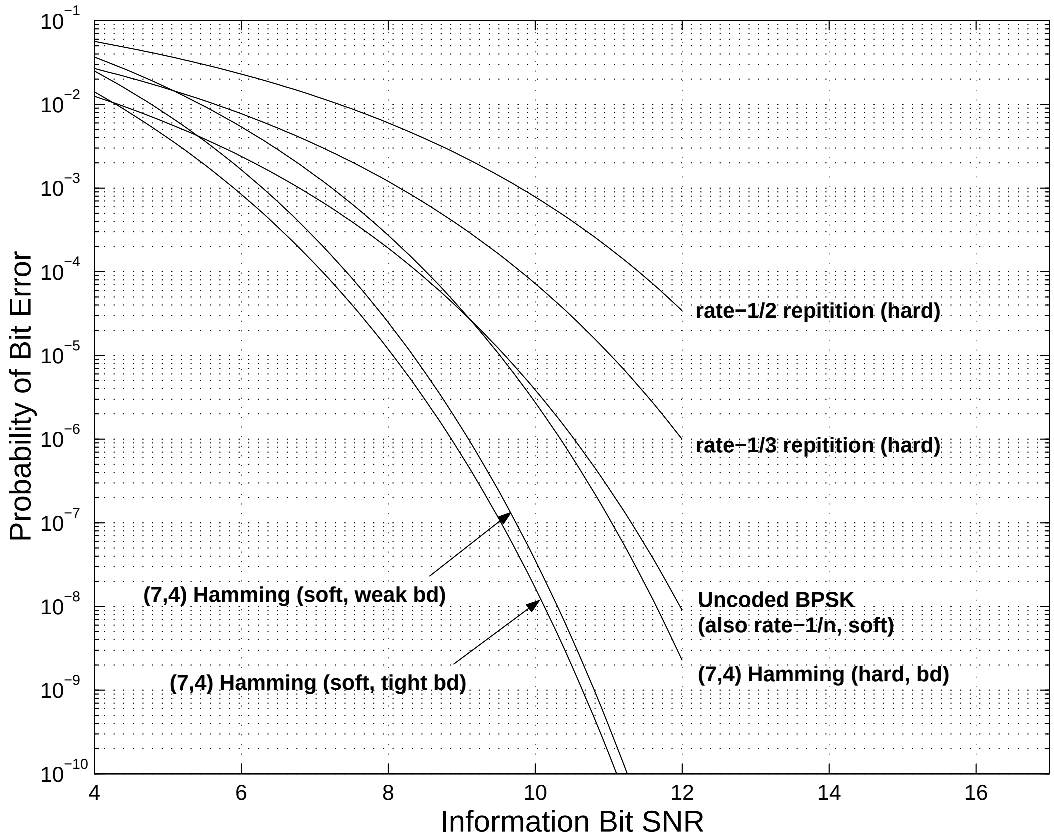

Figure 1: Performance of block codes for BPSK in AWGN

- ∘ For hard decision decoding, $\mathrm{sgn}(y_{\ell,I})$, $\ell = 1, 2, \ldots, n$, are sent to the decoder. This is suboptimum but is used in practice for block codes since optimum (soft decision) decoding is very complex for large n, and efficient hard-decision decoders can be designed for many good block codes. For convolutional codes, there is no reason to resort to hard decision decoding, since soft decision decoding can be done without significantly increased complexity.

- ∘ For optimum (soft decision) decoding, $\{y_\ell\}_{\ell=1}^n$ are sent directly to the decoder for optimum decoding of the codewords.

- ∘ *MPE (ML) Decoding:* Let $p_j(\mathbf{y})$ denote the conditional pdf of $\mathbf{y}$, given $\mathbf{c}_j$ that is transmitted. Then, assuming all codewords are equally likely to be transmitted, the MPE estimate of the transmitted codeword index is given by

$$\hat{i}_{\text{MPE}} = \hat{i}_{\text{ML}} = \arg\max_j \; p_j(\mathbf{y})\,. \tag{19}$$

○ The MPE estimate of the codeword index is given by

$$
\begin{aligned}
\hat{i}_{\text{MPE}} &= \arg\max_j \; p_j(\mathbf{y}) \\
&= \arg\max_j \prod_{\ell=1}^{n} \frac{1}{\pi N_0} \; \exp\left[-\frac{|y_\ell - (2c_{j,\ell}-1)\sqrt{\mathcal{E}_{\text{c}}}|^2}{N_0} \right] \\
&= \arg\max_j \sum_{\ell=1}^{n} y_{\ell,I} \, c_{j,\ell} \qquad (20)
\end{aligned}
$$

○ If we restrict attention to linear block codes, we can assume that the "all-zeros" codeword is part of the code book. Without loss of generality, we can set $\mathbf{c}_1 = [0\,0\cdots 0]$. Furthermore linear block codes have the symmetry property that the probability of codeword error, conditioned on $\mathbf{c}_i$ being transmitted, is the same for all i.

○ Thus

$$
\begin{aligned}
\text{P}_{\text{ce}} &= \mathsf{P}\left(\{\hat{i} \neq 1\} | \{i=1\}\right) = \mathsf{P}\left(\bigcup_{j=2}^{2^k} \{\hat{i} \neq j\} \Bigg| \{i=1\} \right) \\
&= \mathsf{P}\left(\bigcup_{j=2}^{2^k} \left\{ \sum_{\ell=1}^{n} c_{j,\ell}\, y_{\ell,I} > 0 \right\} \Bigg| \{i=1\} \right) \\
&\leq \sum_{j=2}^{2^k} \mathsf{P}\left(\left\{ \sum_{\ell=1}^{n} c_{j,\ell}\, y_{\ell,I} > 0 \right\} \Bigg| \{i=1\} \right) \qquad (21)
\end{aligned}
$$

where the last line follows from the Union Bound.

○ Now when $\{i=1\}$, i.e., if $\mathbf{c}_1 = \mathbf{0}$ is sent, then

$$
y_{\ell,I} = -\sqrt{\mathcal{E}_{\text{c}}} + w_{\ell,I} \; . \qquad (22)
$$

Thus

$$
\begin{aligned}
\mathsf{P}\left(\left\{ \sum_{\ell=1}^{n} c_{j,\ell}\, y_{\ell,I} > 0 \right\} \Bigg| \{i=1\} \right) &= \mathsf{P}\left\{ \sum_{\ell=1}^{n} c_{j,\ell}\, (-\sqrt{\mathcal{E}_{\text{c}}} + w_{\ell,I}) > 0 \right\} \\
&= \mathsf{P}\left\{ \sum_{\ell=1}^{n} c_{j,\ell} w_{\ell,I} \; > \; \sqrt{\mathcal{E}_{\text{c}}} \sum_{\ell=1}^{n} c_{j,\ell} \right\} \\
&= Q\left(\sqrt{ \frac{2\mathcal{E}_{\text{c}}}{N_0} \sum_{\ell=1}^{n} c_{j,\ell} } \right) \qquad (23)
\end{aligned}
$$

where the last line follows from the fact that $c_{j,\ell}^2 = c_{j,\ell}$.

○ *Definition:* The Hamming weight ω_i of a codeword $\mathbf{c}_i$ is the number of 1's in the codeword, i.e.,

$$
\omega_i = \sum_{\ell} c_{j,\ell} \; . \qquad (24)
$$

◦ Thus

$$P_b \le P_{ce} \le \sum_{j=2}^{2^k} Q(\sqrt{2\,\gamma_c\,\omega_j}\,) = \sum_{j=2}^{2^k} Q(\sqrt{2\,R\,\gamma_b\,\omega_j}\,)\,. \tag{25}$$

◦ To compute the bound on P_b we need the weight distribution of the code. For example, for the (7,4) Hamming code, it can be shown that there are 7 codewords of weight 3, 7 of weight 4, and 1 of weight 7.

◦ We can obtain a weaker bound on P_b using only the minimum distance d_{min} of the code as

$$P_b \le P_{ce} \le \sum_{j=2}^{2^k} Q(\sqrt{2\,R\,\gamma_b\,d_{min}}\,) = (2^k - 1)\,Q(\sqrt{2\,R\,\gamma_b\,d_{min}}\,) \tag{26}$$

◦ *Example:* Rate-$\frac{1}{n}$ repitition code

This code has only one non-zero codeword with weight equal to n. With only one term in the Union Bound, the bound (25) becomes an equality. Furthermore, $P_b = P_{ce}$ in this special case. Thus

$$P_b = P_{ce} = Q\left(\sqrt{2\,\frac{1}{n}\,\gamma_b\,n}\right) = Q(\sqrt{2\,\gamma_b}\,)\,. \tag{27}$$

This means that repitition codes with soft decision decoding have zero coding gain for AWGN channels. (We will see in the next section that repitition codes can indeed provide gains for *fading* channels.)

◦ *Example:* $(7,4)$-Hamming code

Using the weight distribution given above, we immediately obtain:

$$P_b \le 7Q\left(\sqrt{\frac{32}{7}\,\gamma_b}\right) + 7Q\left(\sqrt{\frac{24}{7}\,\gamma_b}\right) + Q\left(\sqrt{8\,\gamma_b}\right)\,. \tag{28}$$

We can also obtain the following weaker bound based on (26):

$$P_b \le 15\,Q\left(\sqrt{\frac{24}{7}\,\gamma_b}\right)\,. \tag{29}$$

See Figure 1 for the performance curves for soft decision making for the repitition and (7,4) Hamming codes. We can see that soft decision decoding improves performance by about 2 dB over hard decision decoding for the (7,4) Hamming code.

Coding and Interleaving for Slow, Flat Fading Channels

⋄ If we send the n bits of the codeword directly on the fading channel, they will fade together if the fading is slow compared to the code bit rate. This results in bursts of errors over the block length of the code. If the burst is longer than the error correcting capability of the code, we have a decoding error.

⋄ To avoid bursts of errors, we need to guarantee that the n bits of the codeword fade independently. One way to do this is via *interleaving* and *de-interleaving*.

⋄ The interleaver follows the encoder and rearranges the output bits of the encoder so that the code bits of a codeword are separated by N bits, where NT_c is chosen to be much greater than the coherence time T_{coh} of the channel.

⋄ The de-interleaver follows the demodulator (and precedes the decoder) and simply inverts the operation of the interleaver so that the code bits of the codeword are back together for decoding. For hard decision decoding the de-interleaver receives a sequence of bit estimates from the demodulator, and for soft decision decoding, the decoder receives a sequence of (quantized) matched filter outputs.

⋄ In the following, we assume perfect interleaving and de-interleaving, so that the bits of the codeword fade independently.

Coded BPSK on Rayleigh fading channel – Hard decision decoding

⋄ The received signal corresponding to the codeword $\mathbf{c}_i$ received in AWGN with independent fading on the bits is:

$$y(t) = \sum_{\ell=1}^{n} \alpha_\ell \, e^{\phi_\ell} \, \sqrt{\mathcal{E}_\text{c}} \; (2c_{i,\ell} - 1) \; g_\ell(t) + w(t) \tag{30}$$

where $\{g_\ell(t)\}_{\ell=1}^n$ are shifted versions of the pulse shaping function $g(t)$ corresponding to the appropriate locations in time after interleaving.

⋄ Assuming that $\{\phi_\ell\}$ are known at the receiver, the output of the matched filter for the ℓ-th code bit interval is given by:

$$y_\ell = (2c_{i,\ell} - 1)\alpha_\ell \, \sqrt{\mathcal{E}_\text{c}} + w_\ell \tag{31}$$

where $\{w_\ell\}$ are i.i.d. $\mathcal{CN}(0, N_0)$.

⋄ For hard decision decoding, we send sgn$(y_{\ell,I})$ to the decoder. Note that we do *not* need to know the fade levels $\{\alpha_\ell\}$ to make hard decisions. However, the error probability corresponding to the hard decisions is a function of the fade levels.

⋄ The conditional code bit error probability is given by:

$$\mathsf{P}(\{\ell\text{-th code bit is in error}\} \mid \alpha_\ell) = Q(\sqrt{2\gamma_{\text{c},\ell}}\,) \tag{32}$$

where

$$\gamma_{\text{c},\ell} = \frac{\alpha_\ell^2 \mathcal{E}_\text{c}}{N_0} = \alpha_\ell^2 \overline{\gamma}_\text{c} \,. \tag{33}$$

⋄ For Rayleigh fading, $\{\gamma_{\text{c},\ell}\}_{\ell=1}^n$ are i.i.d. exponential with mean $\overline{\gamma}_\text{c} = R\overline{\gamma}_\text{b}$.

⋄ Using this fact, we can show that for a t-error correcting code the average probability of codeword error $\overline{\mathsf{P}}_{\text{c e}}$

is given by:

$$\overline{\mathsf{P}}_{\mathrm{c\,e}} \le \sum_{q=t+1}^{n} \binom{n}{q} \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{c}}}{1+\overline{\gamma}_{\mathrm{c}}}}\right)^{q} \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{c}}}{1+\overline{\gamma}_{\mathrm{c}}}}\right)^{n-q} \tag{34}$$

and $\overline{\mathsf{P}}_{\mathrm{b}} \le \overline{\mathsf{P}}_{\mathrm{c\,e}}$ in general. (See Problem 4 of HW#5.)

◇ For a rate-$\frac{1}{n}$ repitition code, it is easy to show that

$$\overline{\mathsf{P}}_{\mathrm{b}} = \overline{\mathsf{P}}_{\mathrm{c\,e}} = \begin{cases} \sum_{q=\frac{n+1}{2}}^{n} \binom{n}{q} \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{n+\overline{\gamma}_{\mathrm{b}}}}\right)^{q} \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{n+\overline{\gamma}_{\mathrm{b}}}}\right)^{n-q} & \text{if } n \text{ is odd} \\ \frac{1}{2}\binom{n}{\frac{n}{2}}\left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{n+\overline{\gamma}_{\mathrm{b}}}}\right)^{\frac{n}{2}} \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{n+\overline{\gamma}_{\mathrm{b}}}}\right)^{\frac{n}{2}} & \\ \quad + \sum_{q=\frac{n}{2}+1}^{n} \binom{n}{q} \left(\frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{n+\overline{\gamma}_{\mathrm{b}}}}\right)^{q} \left(\frac{1}{2} + \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{n+\overline{\gamma}_{\mathrm{b}}}}\right)^{n-q} & \text{if } n \text{ is even} \end{cases} \tag{35}$$

In the special case of a rate-$\frac{1}{2}$ repitition code we obtain:

$$\overline{\mathsf{P}}_{\mathrm{b}} = \overline{\mathsf{P}}_{\mathrm{c\,e}} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}_{\mathrm{b}}}{2+\overline{\gamma}_{\mathrm{b}}}} \tag{36}$$

which is 3 dB worse than the error probability without coding.

Coded BPSK on Rayleigh fading channel – Soft decision decoding

◇ Assuming that $\{\phi_\ell\}$ are known at the receiver, the output of the matched filter for the ℓ-th code bit interval is given by:

$$y_\ell = (2c_{i,\ell} - 1)\alpha_\ell \sqrt{\mathcal{E}_{\mathrm{c}}} + w_\ell \tag{37}$$

where $\{w_\ell\}$ are i.i.d. $\mathcal{CN}(0, N_0)$.

◇ MPE decoding (for fixed and known $\{\alpha_\ell\}$): We follow the same steps as in the pure AWGN case to get

$$\begin{aligned} \hat{i}_{\mathrm{MPE}} &= \arg\max_j \prod_{\ell=1}^{n} \frac{1}{\pi N_0} \exp\left[-\frac{|y_{\ell,I} - (2c_{j,\ell}-1)\alpha_\ell\sqrt{\mathcal{E}_{\mathrm{c}}}\,|^2}{N_0}\right] \\ &= \arg\max_j \sum_{\ell=1}^{n} \alpha_\ell\, y_{\ell I}\, c_{j,\ell} \end{aligned} \tag{38}$$

Note that, unlike in hard decision decoding, we need to know the fade levels $\{\alpha_\ell\}$ for soft decision decoding.

◇ Again, as in pure AWGN case, we can set $\mathbf{c}_1 = \mathbf{0}$ and compute a bound on P_{ce} for fixed $\{\alpha_\ell\}$ as:

$$\mathsf{P}_{\mathrm{ce}}(\alpha_1, \ldots, \alpha_n) \le \sum_{j=2}^{2^k} \mathsf{P}\left(\left\{\sum_{\ell=1}^{n} \alpha_\ell\, c_{j,\ell}\, y_{\ell I} > 0\right\} \middle| \{i=1\}\right) = Q\left(\sqrt{2\sum_{\ell=1}^{n} c_{j,\ell}\, \gamma_{\mathrm{c},\ell}}\right) . \tag{39}$$

◇ Let

$$\beta_j = \sum_{\ell=1}^{n} c_{j,\ell}\, \gamma_{c,\ell}\,. \tag{40}$$

Then β_j is the sum of ω_j i.i.d. exponential random variables, where ω_j is the weight of $\mathbf{c}_j$. Thus

$$p_{\beta_j}(x) = \left(\frac{1}{\overline{\gamma}_c}\right)^{w_j} \frac{x^{\omega_j - 1}}{(\omega_j - 1)!} \exp\left(-\frac{x}{\overline{\gamma}_c}\right) 1\!\!1_{\{x \geq 0\}} \tag{41}$$

◇ Thus the average codeword error probability (averaged over the distribution of the $\{\alpha_\ell\}$) is given by:

$$\overline{\mathsf{P}}_{\mathsf{c\,e}} \leq \sum_{j=2}^{2^k} \int_0^\infty Q(\sqrt{2x}\,) p_{\beta_j}(x)\, dx \tag{42}$$

and clearly $\overline{\mathsf{P}}_{\mathsf{b}} \leq \overline{\mathsf{P}}_{\mathsf{c\,e}}$.

It is easy to show that (see Problem 5 of HW#5):

$$\overline{\mathsf{P}}_{\mathsf{b}} \leq \overline{\mathsf{P}}_{\mathsf{c\,e}} \leq \sum_{j=2}^{2^k} \left(\frac{1-\tilde{\sigma}}{2}\right)^{\omega_j} \sum_{q=0}^{\omega_j - 1} \binom{\omega_j - 1 + q}{q} \left(\frac{1+\tilde{\sigma}}{2}\right)^q \tag{43}$$

where

$$\tilde{\sigma} = \sqrt{\frac{\overline{\gamma}_c}{\overline{\gamma}_c + 1}} = \sqrt{\frac{R\,\overline{\gamma}_b}{R\,\overline{\gamma}_b + 1}}\,. \tag{44}$$

◇ Also, since $\int_0^\infty Q(\sqrt{2x}\,) p_{\beta_j}(x) dx$ decreases as ω_j increases, we have the following weaker bound on $\overline{\mathsf{P}}_{\mathsf{c\,e}}$ in terms of d_{min}.

$$\overline{\mathsf{P}}_{\mathsf{b}} \leq \overline{\mathsf{P}}_{\mathsf{c\,e}} \leq (2^k - 1) \left(\frac{1-\tilde{\sigma}}{2}\right)^{d_{\text{min}}} \sum_{q=0}^{d_{\text{min}} - 1} \binom{d_{\text{min}} - 1 + q}{q} \left(\frac{1+\tilde{\sigma}}{2}\right)^q . \tag{45}$$

◇ For large SNR, i.e., $\overline{\gamma}_c \gg 1$, we have

$$\frac{1+\tilde{\sigma}}{2} \approx 1\,, \quad \text{and} \quad \frac{1-\tilde{\sigma}}{2} \approx \frac{1}{4\,\overline{\gamma}_c} = \frac{1}{4\,R\,\overline{\gamma}_b} \tag{46}$$

and hence the bound in (45) can be approximated by

$$\text{Bound} \approx (2^k - 1) \left(\frac{1}{4\,R\,\overline{\gamma}_b}\right)^{d_{\text{min}}} \binom{2d_{\text{min}} - 1}{d_{\text{min}}} . \tag{47}$$

This means that $\overline{\mathsf{P}}_{\mathsf{b}}$ decreases as $(\overline{\gamma}_b)^{-d_{\text{min}}}$ for large SNR.

◇ *Example:* Rate-$\frac{1}{n}$ repitition code

$$\overline{\mathsf{P}}_{\mathsf{b}} = \overline{\mathsf{P}}_{\mathsf{c\,e}} = \left(\frac{1-\tilde{\sigma}}{2}\right)^n \sum_{q=0}^{n-1} \binom{n - 1 + q}{q} \left(\frac{1+\tilde{\sigma}}{2}\right)^q \tag{48}$$

with $\tilde{\sigma} = \sqrt{\overline{\gamma}_b/(\overline{\gamma}_b + n)}$

The average bit error probability is the same as that obtained with n-th order diversity and maximum ratio combining (as expected).

⋄ *Example:* $(7, 4)$-Hamming code

$$\overline{P}_b \le \overline{P}_{c\,e} \le 7f(3) + 7f(4) + f(7) \tag{49}$$

where

$$f(\omega) = \left(\frac{1-\tilde{\sigma}}{2}\right)^{\omega} \sum_{q=0}^{\omega-1} \binom{\omega-1+q}{q} \left(\frac{1+\tilde{\sigma}}{2}\right)^{q} \tag{50}$$

and $\tilde{\sigma} = \sqrt{4\overline{\gamma}_b/(4\overline{\gamma}_b + 7)}$

⋄ Performance plots for these codes for both hard and soft decision making are shown in Figure 2.

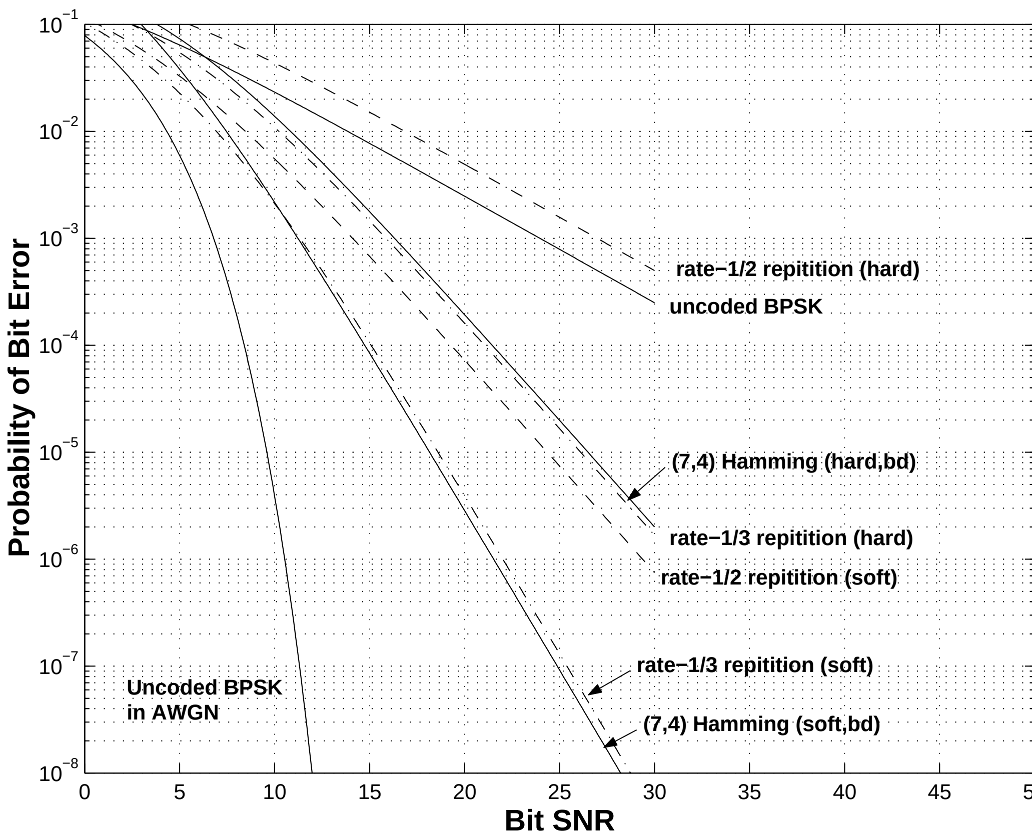

Figure 2: Performance of block codes for BPSK in Rayleigh fading with perfect interleaving