ECE 559: Wireless Communication

Project Report

Diversity – Multiplexing Tradeoff in MIMO Channels with partial CSIT

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1. Summary

In this project, I have studied the performance gain of MIMO systems. There are two types of gain provided by the MIMO channels: one measures the increase of the capacity by the number of *degrees of freedom* of the channel (*spatial multiplexing gain*), and the other improvement of the reliability by the *diversity gain*. Most of previous works focus on either one of these two types of gain by designing schemes to extract either maximal diversity gain or maximal spatial multiplexing gain. However, maximizing one type of gains may not necessarily maximize the other. This make it difficult to compare diversity-based schemes and multiplexing-based schemes. Recently, Zheng and Tse [1] proposed a new point of view that unifies these two types of gains: given a MIMO channel, both gains can be simultaneously obtained, but there is a tradeoff between how much of each type of gain any coding scheme can extract. The optimal diversity-multiplexing tradeoff is a fundamental performance limit of a MIMO channel.

By studying this fundamental tradeoff, we can gain important insights on the resource constraint of MIMO systems, as well as on specific schemes to communicate over such systems. The tradeoff curve can be used as a unified framework to compare the performance of various space-time coding schemes for MIMO channels and stimulates the design of new schemes. For each scheme, we can compute the achievable diversity-multiplexing tradeoff curve and compare it against the optimal curve. By doing this, we can take into account both the capability of a scheme to combat against fading (through diversity gain) and its achieved spatial multiplexing gain. Therefore, the tradeoff curve provides a unified framework to make fair comparisons.

The analysis focus on slow fading channel (the channel is constant during a block) and high SNR region. The approach used for both cases of no CSIT and partial CSIT based on evaluating the outage probability at high SNR.

For the case of no CSI at the transmitter, the optimal tradeoff curve is a piecewise linear curve connecting the points (r, $(m-r)(n-r)$), where r is the multiplexing gain and $(m-r)(n-r)$ is the diversity gain. Intuitively, we can think of a point on the optimal curve as followed: If we want to use r transmit antennas and r receive antennas for multiplexing

gain to increase the rate, the remaining (*m*-*r*) transmit antennas and (*n*-*r*) receive antennas can be used to give the diversity gain of (*m*-*r*) (*n*-*r*).

Consider an example of the 2x2 system. The maximal diversity gain for this channel is $d^*_{\max} = 4$, and the total number of degrees of freedom in the channel is $r^*_{\max} = 2$. To get the maximal diversity gain, each information bit needs to pass through all the four paths from the transmitter to the receivers. The simplest way is to use repetition code, this will guarantee the maximal diversity, however, it only achieved the rate 1/2 since only one symbol is transmitted over two symbol periods. As in class we see that, the Alamouti scheme can also achieve the full diversity. However in terms of tradeoff achieved, the Alamouti scheme is strictly better than the repition scheme, since it achieves a higher diversity gain for any posititve spatial multiplexing gain and its maximal multiplexing gain is 1. However, the tradeoff curve achieved by the Alamouti scheme is still below the optimal curve for positive values of *r*. Figure 1 shows an example of different coding schemes for the 2x2 MIMO channel.

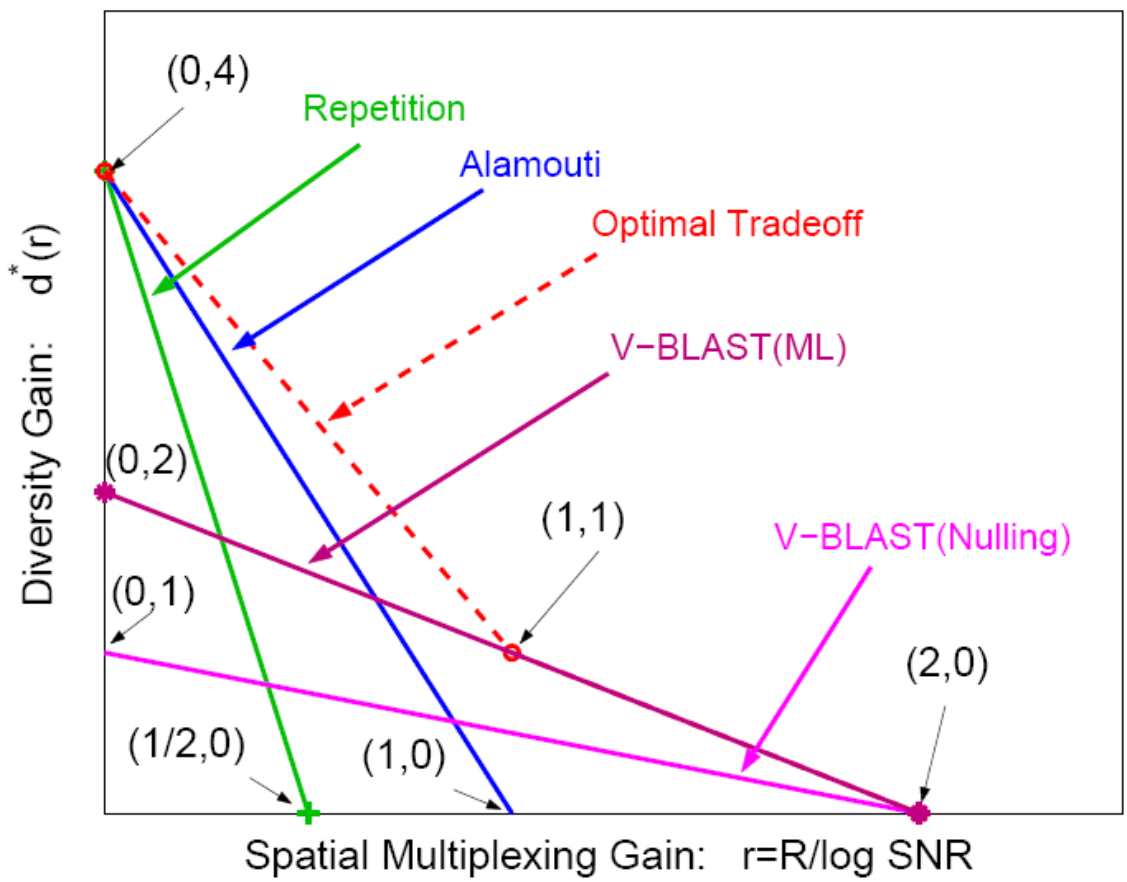

Figure 1. Example of tradeoff curves for different coding schemes

For the case of partial CSIT, the receiver, after estimating the channel will feedback an index to the transmitter. This index indicates the region to which the channel state belongs. Based on this information, the transmitter will control the power to minimize the outage probability while remain sastifying the long term power constraint. The tradeoff curve derivation for this case is similar to the case when there is no CSI at transmitter which is based on the outage probability. The important point in this case is finding the optimal index mapping for different states of the channel. Another important point is the concept of extended approximately universal condition (stated in the slide) which guarantees the upper bound and lower bound to be matched. The main results are given in the attached slide. There are two interesting results:

- The maximal diversity gain of MIMO channel can be increased exponentially with the number of quantized regions ($d^*(0) = \sum_{k=1}^{K} (mn)^k$, where K is the number of quantized regions). Note that when K = 1 (no feedback), the result is the same as before. This result suggests that, we can use only a few bit of feedback to get the diversity gain and save most of antenna resource for multiplexing gain.
- When the adaptive rate transmission scheme is used (different multiplexing gains r are used for different regions of the channel state, we can achieve non-zero diversity gain at maximal multiplexing gain. This can not be achieved in the case of no CSIT. For K = 2, the full diversity gain of the no-CSIT case (mn) can be achieved at maximal multiplexing gain.

The next section provides some detailed proofs stated in the presentation to derive the optimal tradeoff curve of the MIMO channel when the transmitter does not know the CSI. The approach to deriving the tradeoff curve for the partial CSIT case is the same except the introduction of extended approximately universal condition and the optimal index mapping scheme which results in the much higher diversity gain. The detailed of the proofs can be found in [2], which I will not cover here.

2. Diversity-Multiplexing tradeoff with no CSI at transmitter [1]

This section gives some detailed derivation of the optimal diversity-multiplexing tradeoff curve. As we know from the presentation, the approach is as followed:

- Formulate the outage probability
- Lower bound the error probability by outage probability
- Find the upper bound on the error probability
- The upper bound turns out to match the lower bound, therefore we get the SNR exponent of the error probability

Definitions: A scheme is said to achieve spatial multiplexing gain r and the diversity gain d if the data rate:

$$\lim_{SNR\to\infty}\frac{R(SNR)}{\log SNR}=r,\qquad \text{denoted by } R\doteq r\log SNR$$

and the average error probability

$$\lim_{SNR\to\infty}\frac{\log P_e(SNR)}{\log SNR}=-d,\qquad \text{denoted by } P_e\doteq SNR^{-d}$$

2.1 Outage probability

As in the presentation we have, on the scale of interest (high SNR):

$$P_{out}(R)\doteq P[\log\det(I+SNR\mathbf{HH}^{\dagger})<R]$$

To compute the asymptotical outage probability, we need the following lemma:
Lemma 1: let ***R*** be an *m* x *n* random matrix with i.i.d *CN*(0,1) entries. Suppose $m\ge n,\mu_1\le\mu_2\le\ldots\le\mu_1$ be the ordered non-zero eigenvalues of $\mathbf{R}^{\dagger}\mathbf{R}$, then the joint pdf of μ_i's is:

$$p(\mu_1,\ldots,\mu_n)=K_{m,n}^{-1}\prod_{i=1}^{n}\mu_i^{m-n}\prod_{i<j}(\mu_i-\mu_j)^2e^{-\sum_i\mu_i}$$

Where $K_{m,n}$ is the normalizing constant. Define: $\alpha_i\triangleq-\log\mu_1/\log SNR$, for all i. Then we can find the joint pdf of α_i's:

$$p(\alpha)=K_{m,n}^{-1}(\log SNR)^n\prod_{i=1}^{n}SNR^{-(m-n+1)\alpha_i}.\prod_{i<j}(SNR^{-\alpha_i}-SNR^{-\alpha_j})^2\exp[-\sum_{i=1}^{n}SNR^{-\alpha_i}]$$

Now consider the outage probability with $R=r\log SNR$, let $\lambda_1\le\lambda_2\le\ldots\le\lambda_n$ be the non-zero eigenvalues of $\mathbf{HH}^{\dagger}$, and $\lambda_i=SNR^{-\alpha_i}$, we have:

$$P_{out}(R)\doteq P[\log\det(I+SNR\mathbf{HH}^{\dagger})<R]=P[\prod_{i=1}^{n}(1+SNR\lambda_i)<SNR^r]$$

At high SNR, we have $(1+SNR\lambda_i)\doteq SNR^{(1-\alpha_i)^+}$, where $(x)^+$ denotes max{x,0}. Then,

$$P_{out}(R)\doteq P[\prod_i SNR^{(1-\alpha_i)^+}<R]=P[\sum_i(1-\alpha_i)^+<r]$$

Since, we are only interested in the SNR exponent of P_{out}, we can ignore the term $K_{m,n}^{-1}(\log SNR)^n$. Furthermore, for any $\alpha_i<0$, the term $(-SNR^{-\alpha_i})$ decays exponentially

with SNR. Thus, we can ignore the integral over the range with $\alpha_i < 0$, and define the set $A' = \{\alpha \in \Re^{\min\{m,n\}+} \mid \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{\min\{m,n\}} \geq 0,$ and $\sum_{\mathrm{i}} (1-\alpha_i)^+ < r\}$, then:

$$P_{out}(r \log SNR) \doteq \int_{A'} \prod_{i=1}^{n} SNR^{-(m-n+1)\alpha_i} . \prod_{i<j} (SNR^{-\alpha_i} - SNR^{\alpha_j})^2 d\alpha$$

Then, we have the following theorem:

Theorem 1: Let the data rate $R = r \log SNR$, with $r \leq \min\{m,n\}$. The outage probability sastifies:

$$P_{out}(r \log SNR) \doteq SNR^{-d_{out}(r)}$$

$$\text{where } d_{out}(r) = \inf_{\alpha \in A'} \sum_{i=1}^{\min\{m,n\}} 2i - 1 + |m-n| \alpha_i$$

Proof. See [1] for detailed proof.

The minimizing α_i^* can be explicitly computed. In the case that *r* takes an interger value *k*, we have: $\alpha_i^* = 1,$ for $i = 1,...,n-k$ and $\alpha_i^* = 0,$ for $i = n-k+1,...,n$.
Intuitively, in a general *m*x*n* system, an outage occurs when the channel matrix **H** is "near singular". The key step is to explicitly quantify how singular **H** needs to be for outage to occur. Since the smaller singular values have a much higher probability to be close to zero than the larger ones, the typical outage event has *n*-*k* smallest singular values $\lambda_i \doteq SNR^{-1}$ and *k* largest singular values are of order 1.
For the case that *r* is not an interger, say $r \in (k, k+1)$, we have:

$$\alpha_i^* = 1, \text{ for } i = 1,...,n-k-1, \quad \alpha_i^* = 0, \text{ for } i = n-k+1,...,n \text{ and } \alpha_{n-k} = k+1-r .$$

With these optimized value of α_i^*, we get

$$d_{out}^*(r) = (m-r)(n-r)$$

2.2 Error Probability

Lower Bound:
First, we will prove the error probability is lower bound by the outage probability.
Fix a codebook *C* of size 2^{Rl}, and let $\mathbf{X} \in C^{mxl}$ be the input of the channel, which is uniformly drawn from the codebook *C*. Conditioned on a specific channel realization **H** = H, by Fano's inequality, we have:

$$Rl \le 1 + P(\text{error} \mid \mathbf{H} = H)Rl + I(\mathbf{X};\mathbf{Y} \mid \mathbf{H} = H), \text{ hence}$$
$$P(\text{error} \mid \mathbf{H} = H) \ge 1 - \frac{I(\mathbf{X};\mathbf{Y} \mid \mathbf{H} = H)}{lr \log SNR} - \frac{1}{lr \log SNR}$$

The last term goes to zero as SNR goes to infinity. Averaging over **H** gives the average error probability

$$P_e(SNR) = E_{\mathbf{H}}[P(\text{error} \mid \mathbf{H} = H)]$$

Define a set $D_\delta \triangleq \{H : I(\mathbf{X};\mathbf{Y} \mid \mathbf{H} = H) < (r-\delta) l \log SNR\}$, for any $H \in D_\delta$, the probability of error is lower bounded by $1 - \frac{r-\delta}{r} + o(1)$, hence

$$P_e(SNR) \ge \left(1 - \frac{r-\delta}{r} + o(1)\right) P(D_\delta)$$

Then choose the input **X** to minimize $P(D_\delta)$ and apply the theorem 1, we have:

$$\begin{aligned} P_e(SNR) &\dot{\ge} \left(1 - \frac{r-\delta}{r} + o(1)\right) SNR^{-d_{out}(r-\delta)} \\ &\doteq SNR^{-d_{out}(r-\delta)} \end{aligned}$$

Take $\delta \to 0$, by the continuity of $d_{\text{out}}(r)$, we have the lower bound $P_e(SNR) \dot{\ge} SNR^{-d_{out}(r)}$
This result suggests that conditioned on the channel outage event, it is very likely that a detection error occurs.

Upper Bound:
Choose the input to be the random code from the i.i.d Gaussian ensemble. We have,

$$\begin{aligned} P_e(SNR) &= P_{out}(R) P(\text{error} \mid \text{outage}) + P(\text{error, no outage}) \\ &\le P_{out}(R) + P(\text{error, no outage}) \end{aligned}$$

The second term can be upper-bounded via a union bound. Assume $X(0)$, $X(1)$ are two possible transmitted codeword and $\Delta X = X(0) - X(1)$. Suppose $X(0)$ is transmitted, the probability that an ML receiver will make a detection error in favor of $X(1)$, conditioned on a certain realization of the channel, is

$$P(X(0) \to X(1) \mid \mathbf{H} = H) = P\left(\frac{SNR}{m} \left\| \frac{1}{2} H(\Delta X) \right\|_F^2 \le \|\mathbf{w}\|^2 \right)$$

Where $\mathbf{w}$ is the additive noise on the direction of $H(\Delta X)$ with variance 1/2. Using the upper bound for the Q-function, we have

$$P(X(0) \to X(1) \mid \mathbf{H} = H) \le \exp\left(-\frac{SNR}{4m} \left\| H(\Delta X) \right\|^2 \right)$$

Averaging over the ensemble of random codes, we have the average pairwise error probability (PEP) conditioned on the channel realization [3]

$$P(X(0) \to X(1) \mid \mathbf{H} = H) \le \det\left(I + \frac{SNR}{2m} HH^{\dagger} \right)^{-l}$$

Apply the union bound for SNR^{lr} codewords, we have

$$\begin{aligned} P(X(0) \to X(1) \mid \mathbf{H} = H) &\le SNR^{lr} \exp\left(-\frac{SNR}{4m} \left\| H(\Delta X) \right\|^2 \right) \\ &= SNR^{lr} \prod_{i=1}^{\min\{m,n\}} (1 + \frac{SNR}{2m} \lambda_i)^{-l} \end{aligned}$$

Averaging with respect to the distribution of the SNR exponent α of singular values and with a similar argument as in Theorem 1, we can approximate this as

$$P(\text{error, no outage}) \dot{\le} \int_{(A')^c} SNR^{-d_G(r,\alpha)} d\alpha$$

$$\text{with } d_G(r,\alpha) = \sum_{i=1}^{\min\{m,n\}} 2i - 1 + |m-n| \alpha_i + l \left(\sum_{i=1}^{\min\{m,n\}} (1-\alpha_i)^+ - r \right)$$

The probability is dominated by the term corresponding to α^* that minimizes $d_G(r,\alpha)$. For $l \ge 2(\min\{m,n\}) - 1 + |m-n| = m+n-1$, the minimum always occurs with $\sum (1-\alpha_i^*) = r$, hence

$$d_G(r) \min_{\sum \alpha_i = \min\{m,n\} - r} \sum_{i=1}^{\min\{m,n\}} (2i - 1 + |m-n|)\alpha_i = d_{out}(r) \quad \forall r$$

Therefore

$$\begin{aligned} P_e(SNR) &\le P_{out}(R) + P(\text{error, no outage}) \\ &\dot{\le} SNR^{-d_{out}(r)} + SNR^{-d_G(r)} \doteq SNR^{-d_{out}(r)} \end{aligned}$$

We see that the upper bound and the lower bound are matched, thus $P_e(SNR) \doteq SNR^{-d_{out}(r)}$, or we have the diversity gain $d_{out}(r) = d^*(r) = (m-r)(n-r)$ which gives the optimal tradeoff curve.

3. Q&A

Q: How can the tradeoff curve be utilized? (What is the application of the tradeoff curve?)

A: The optimal tradeoff curve shows the optimal tradeoff between diversity gain and multiplexing gain provided by the MIMO channel. It is used as a unified framework to compare different space-time codes. By computing the tradeoff curve of each specific scheme and compare it to the optimal curve, we can see how optimal that scheme is. Since some schemes are designed to maximize the capability to combat fading and the others are designed to maximize the transmission rate, it is difficult to compare these schemes. However, by computing the tradeoff curves of these schemes, we take into account both of these capabilities that each schemes can achieve. That make the fair comparison. Beside, the optimal tradeoff curve may stimulate the design of new schemes to utilize the resources given by the MIMO channels. However, it may not show how to achieve the optimal curve.

Q: It is not clear to me if this result provides some bound on general space-time code?

A: Yes. Because, the optimal curve is the upper bound on the diversity gain and multiplexing gain that a MIMO channel can support. Any space-time code is designed to utilized these gains and can not get above the optimal curve.

Q: What is the intuition behind the piecewise linear tradeoff curve? Is it some kind of time multiplexing? And,
Q: How can we achieve points in the curve by time-sharing? Particularly, how to achieve the point (0,1) in Alamouti scheme for 2x1 system?

A: A specific point on the curve is not achieved in the sense of time sharing or time multiplexing. The optimal curve is derived by fixing the multiplexing gain *r*, and find the maximal diversity gain that can be achieved at that rate. Therefore, a specific point on the curve illustrates that if we want to operate the system at the rate $R \doteq r \log SNR$, the maximal diversity we can get is the diversity gain at point (r, $d(r)$).
Intuitively, for a *m*x*n* MIMO system, if we use *r* transmit antennas and *r* receive antennas for multiplexing gain to increase the rate, the remaining (*m*-*r*) transmit antennas and (*n*-*r*) receive antennas can be used to give the diversity gain of (*m*-*r*) (*n*-*r*).
For the Alamouti scheme, the point (0,1) can be achieved by scaling the rate R as $R = \log SNR$ and compute the error probability when SNR goes to infinity, we can see

that the SNR exponent of the error probability is the same as the single antenna case, which give zero diversity gain.

Q: How can a specific point on the tradeoff curve can be achieved in practice?

A: The optimal curve does not tell how to achieve a specific point in practice, since it only gives the bound on the gains that a channel may provide.

Q: Practically speaking, when should we choose between diversity and multiplexing gains? Which should be prefered? Is there a relation between the reliability and the scheme that we choose?

A: In my opinion, this depend on the channel conditions and the application we want to provide. The diversity oriented schemes may be chosen in the case of bad fading channel where we want to provide independent fading paths to combat fading. The multiplexing-oriented schemes may be chosen when we want high data-rate applications but do not have tight requirements on error probablity.

Q: Can random coding be used to arrive at the lower bound on the probability of error?

A: In the outage probability formulation, the input is taken as the Gaussian distribution and then the outage probability is taken as the infimum over all input distribution. Since the noise is Gausian, the Gaussian input maximize the mutual information and the outage occurs when the mutual information drops below the required rate R, this give the lower bound on outage probability for every input distribution. Furthermore, as we showed that the error is very likely to occur when the outage event occurs, therefore we can use outage probability as a lower bound for the error probability.

Q: How is the tradeoff curve affected by feedback from the receiver?

A: When the receiver feedbacks the quantized value of CSI which indicate the region in which the channel appears, to the transmitter, the transmitter can use that information to control the power. The diversity gain achieved is surprisingly increased exponentially with the number of quantized regions, the maximal diversity gain which can be achieved is $\sum_{k=1}^{K}(N_t N_r)^k$, where K is the number of quantized regions [2]. Another interesting result is that, when adaptive rate transmission is used (different multiplexing gains for different channel state regions), we can have non-zero diversity gain at maximal multiplexing gain, which can not be achieved in the case of no CSIT.

References

[1] L. Zheng and D. Tse, "Diversity and Multiplexing: A Fundamental Tradeoff in Multiple Antenna Channels", IEEE Transactions on Information Theory, vol. 49, May 2003, pp. 1073-96.

[2] Tung T. Kim and Mikael Slogkund, "Diversity-Multiplexing Tradeoff in MIMO Channels with Partial CSIT", *IEEE Trans. Inform. Theory*, vol. 53, NO. 8, August 2007.

[3] V.Tarokh, N. Seshadri, and Calderbank, "Space-time codes for high data rate wireless communications: Performance criterion and code construction", *IEEE Trans. Inform. Theory*, vol. 44, pp. 774-765, Mar. 1998