Diversity-Multiplexing Tradeoff in MIMO Channels with Partial CSIT

ECE 559 Presentation

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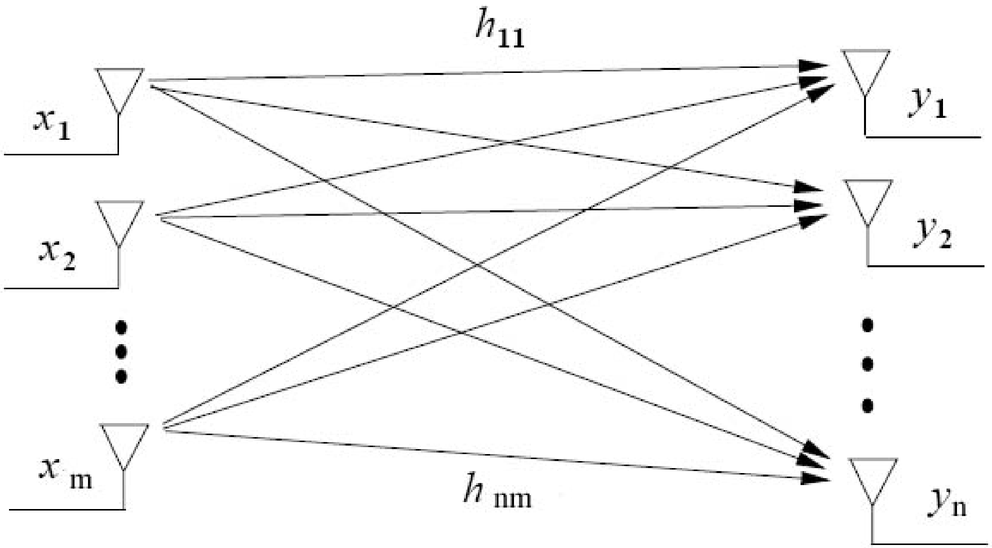
Dec 3, 2007

Introduction

- MIMO systems provide two types of gains
  - Diversity Gain: each path from a transmitter to a receiver experiences independent fading
  - Spatial Multiplexing Gain: fading creates increased degrees of freedom
- There is a fundamental tradeoff between diversity gain and spatial multiplexing gain

No CSIT: Channel Model



$$\mathbf{Y} = \sqrt{\frac{\mathsf{SNR}}{m}}\mathbf{H}\mathbf{X} + \mathbf{W}$$

$$\mathbf{X} \in \mathcal{C}^{m \times l}$$

$$\mathbf{Y} \in \mathcal{C}^{n \times l} \qquad \mathbf{w}_{ij} \sim \mathcal{CN}(0,1)$$

$$\frac{1}{|\mathcal{C}|}\sum_{i=1}^{|\mathcal{C}|} \|X(i)\|_F^2 \leq ml.$$

- H known at receiver, not transmitter, constant within a block of *l* symbols
- Analysis focuses on high SNR regime

Diversity Gain

- Example: Uncoded Binary PSK over single antenna: $P_e(\mathsf{SNR}) \approx \frac{1}{4}\mathsf{SNR}^{-1}.$

- In Contrast: Transmitting the same signal to a receiver with 2 antenna: $P_e(\mathsf{SNR}) \approx \frac{3}{16}\mathsf{SNR}^{-2}.$

- At high SNR, performance gain characterized by SNR exponent of error probability d (Diversity Gain)

$$\lim_{\mathsf{SNR}\to\infty} \frac{\log P_e(\mathsf{SNR})}{\log \mathsf{SNR}} = -d$$

Define as: $P_e(\mathsf{SNR}) \doteq \mathsf{SNR}^{-d}.$

- For m transmit and n receive antennas, max diversity gain = mn

Multiplexing Gain

- Ergodic capacity of the channel we defined, with **H** independently and identically distributed across blocks

$$C(\mathsf{SNR}) = \mathcal{E}\left[\log\det\left(I + \frac{\mathsf{SNR}}{m}\mathbf{H}\mathbf{H}^{\dagger}\right)\right]$$

- At high SNR:

$$C(\mathsf{SNR}) = \min\{m,n\}\log\frac{\mathsf{SNR}}{m} + \sum_{i=|m-n|+1}^{\max\{m,n\}} \mathcal{E}[\log\chi^2_{2i}] + o(1),$$

- Channel capacity increases with min{m, n} logSNR, instead comparision with logSNR for single antenna channels

spatial multiplexing gain of r $\qquad \lim_{\mathsf{SNR}\to\infty} \frac{R(\mathsf{SNR})}{\log\mathsf{SNR}} = r$

- Where we thinks of a family of codes {***C***(SNR)}, one at each SNR level and R(SNR) is the rate of code ***C***(SNR)

Optimal Tradeoff

- For the case

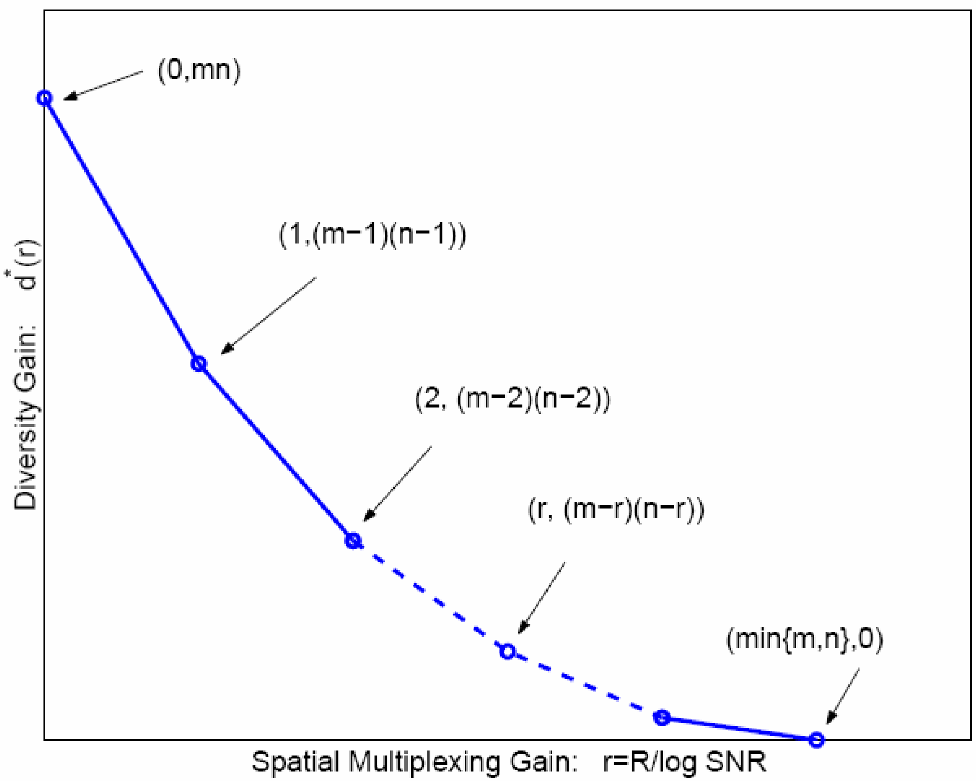
$$l \geq m + n - 1.$$

- Optimal tradeoff curve is a piecewise linear function connecting the points:

$$(k, d^*(k)), k = 0, 1, \ldots, \min\{m, n\}$$

- Where:

$$d^*(k) = (m-k)(n-k)$$



Approach to Proving the Tradeoff Curve for $l \geq m + n - 1$.

- Formulate Outage Probability $P_{out}(r \log \mathsf{SNR}) \doteq \mathsf{SNR}^{-d_{out}(r)}$
  - $d_{out}(r)$ turns out to equal $d^*(r)$
- Lower bound error probability with outage probability

$$P_e(\mathsf{SNR}) \dot{\geq} \mathsf{SNR}^{-d_{out}(r)} \quad \text{for} \quad R = r \log \mathsf{SNR}$$

- Upper bound error probability. It turns out

$$P_e(\mathsf{SNR}) \dot{\leq} \mathsf{SNR}^{-d_{out}(r)}$$

- Therefore $P_e(\mathsf{SNR}) \doteq \mathsf{SNR}^{-d_{out}(r)}$

Outage Probability

- Outage occurs when mutual information is below the data rate

$$\{H : I(\mathbf{x}_t; \mathbf{y}_t \mid \mathbf{H} = H) < R\}$$

- Without loss of optimality, take input to be Gaussian

$$I(\mathbf{x}_t; \mathbf{y}_t \mid \mathbf{H} = H) = \log\det\left(I + \frac{\mathsf{SNR}}{m} HQH^\dagger\right)$$

- Optimizing over all input distributions

$$P_{out}(R) \quad = \quad \inf_{Q \geq 0, \mathrm{trace(Q)} \leq \mathrm{m}} P\left[\log\det\left(I + \frac{\mathsf{SNR}}{m}\mathbf{H}Q\mathbf{H}^\dagger\right) < R\right]$$

- Using a lower and upper bound, and taking SNR to inf.

$$P_{out}(R) \doteq P\left[\log\det\left(I + \mathsf{SNR}\mathbf{H}\mathbf{H}^\dagger\right) < R\right].$$

Outage Probability

- After significant mathematical manipulation, we get

$$P_{out}(r \log \mathsf{SNR}) \doteq \mathsf{SNR}^{-d_{out}(r)}$$

where

$$d_{out}(r) = \inf_{\alpha \in \mathcal{A}'} \sum_{i=1}^{\min\{m,n\}} (2i - 1 + |m-n|)\, \alpha_i$$

$$\mathcal{A}' = \{\alpha \in \mathcal{R}^{\min\{m,n\}+} \mid \alpha_1 \geq \ldots \geq \alpha_{\min\{m,n\}} \geq 0, \text{ and } \sum_i (1-\alpha_i)^+ < r\}$$

- $d_{out}(r)$ can be solved explicitly, and turns out to be equal to d*(r) in optimal trade off curve

Lower Bound on Probability of Error

- Fix a codebook C of size 2^{Rl} and let $\mathbf{X} \in \mathcal{C}^{m \times l}$ be the input, uniformly drawn from C

- For a specific realization **H** = H, applying Fano's inequality and rearranging, we can write, for $R = r \log \mathsf{SNR}$,

$$P(\text{error} \mid \mathbf{H} = H) \geq 1 - \frac{I(\mathbf{X}; \mathbf{Y} \mid \mathbf{H} = H)}{lr \log \mathsf{SNR}} - \frac{1}{lr \log \mathsf{SNR}}$$

- We want an expression for $P_e(\mathsf{SNR})$ as SNR goes to inf.

- Consider scenario:

for any $\delta > 0$, for any H in the set

$$\mathcal{D}_\delta \triangleq \{H : I(\mathbf{X}; \mathbf{Y} \mid \mathbf{H} = H) < (r - \delta) l \log \mathsf{SNR}\}$$

the probability of error is lower bounded by $1 - \frac{r-\delta}{r} + o(1)$;

Lower Bound on Probability of Error

- Therefore, we have

$$P_e(\mathsf{SNR}) \geq \left(1 - \frac{r-\delta}{r} + o(1)\right) P(\mathcal{D}_\delta)$$

- Applying the same method as in deriving outage probability, we can write

$$\begin{aligned} P_e(\mathsf{SNR}) &\dot{\geq} \left(1 - \frac{r-\delta}{r} + o(1)\right) \mathsf{SNR}^{-d_{out}(r-\delta)} \\ &\doteq \mathsf{SNR}^{-d_{out}(r-\delta)} \end{aligned}$$

- Take δ to the limit of 0, and by continuity of $d_{out}(r)$, we get

$$P_e(\mathsf{SNR}) \dot{\geq} \mathsf{SNR}^{-d_{out}(r)}$$

- Suggests: if channel is in outage, error very likely

Upper Bound on Error Probability

- Choose input to be random code from i.i.d Gaussian ensemble
- We have $$P_e(\mathsf{SNR}) \leq P_{out}(R) + P(\text{ error, no outage})$$
- Use union bound to bound the second term, by considering pairwise error probability, we get

$$P(\text{ error, no outage}) \dot{\leq} \mathsf{SNR}^{-d_G(r)}$$ where $$d_G(r) = d_{out}(r), \forall r.$$

- Putting the two together

$$\begin{aligned} P_e(\mathsf{SNR}) &= P_{outage}(R) + P(\text{ error, no outage}) \\ &\doteq \mathsf{SNR}^{-d_{out}(r)} + P(\text{ error, no outage}) \\ &\dot{\leq} \mathsf{SNR}^{-d_{out}(r)} + \mathsf{SNR}^{-d_G(r)} \\ &\doteq \mathsf{SNR}^{-d_{out}(r)} \end{aligned}$$

Optimal Tradeoff

- Upper and lower bounds match, therefore

$$P_e(\mathsf{SNR}) \doteq \mathsf{SNR}^{-d_{out}(r)}$$

at some rate $R = r \log \mathsf{SNR},$

- Moreover, $d_{out}(r) = d^*(r)$

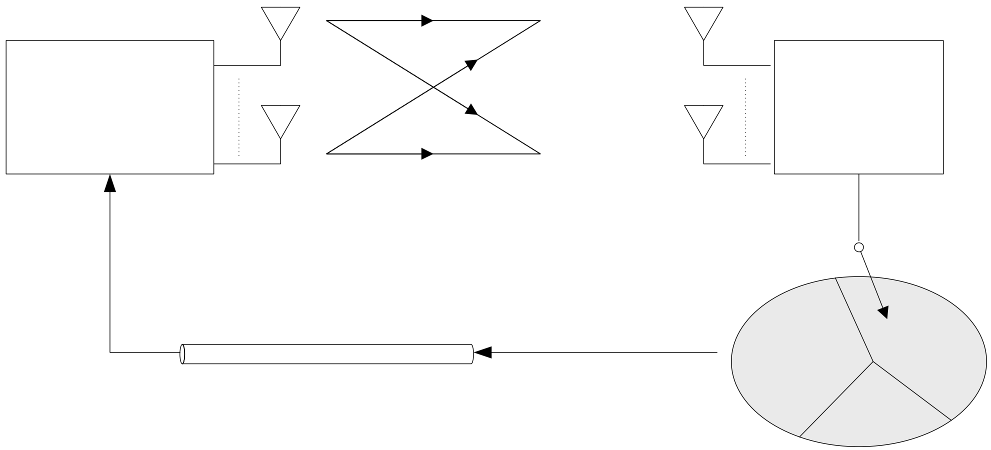
- Therefore, for multiplexing gain r, we get diversity gain d*(r)

- And the optimal tradeoff curve d*(r) is a piecewise linear function connecting

$$(k, d^*(k)), k = 0, 1, \ldots, \min\{m, n\}$$

$$d^*(k) = (m-k)(n-k)$$

Diversity-Multiplexing Tradeoff: Partial CSIT



- System model: During fading block *l*

$$\mathbf{Y}_l = \mathbf{H}_l \mathbf{S}_l + \mathbf{W}_l$$

$$m \triangleq \max(N_r, N_t)$$

$$n \triangleq \min(N_r, N_t)$$

Beamforming

- Conditioned on index $\mathrm{I}(\mathbf{H}_l) = i$ where $i \in \{1, 2, \ldots K\}$, the codeword **S** is taken from a codebook $C_i = \{\mathbf{S}_i(1), \ldots, \mathbf{S}_i(M_i)\}$ with rate $R_i = r_i \log \mathrm{SNR}$

$\mathbf{H}$

Cl
Est

- Define: Average total transmit power

$$\mathbf{P}_i \triangleq \frac{1}{TM_i} \sum_{k=1}^{M_i} \|\mathbf{S}_i(k)\|_F^2$$

- Long-term power constraint

$$\lim_{L\to\infty} \frac{1}{L} \sum_{l=1}^{L} \frac{1}{T} \|\mathbf{S}_l\|_F^2 \overset{a.s}{=} E_{\mathbf{H}}\left[P_{I(\mathbf{H})}\right] \le \mathbf{SNR}$$

Index J_n

Feedback Channel
(delay, error free)

Definitions

- Average rate: $R \triangleq \lim_{L\to\infty} \frac{1}{L}\sum_{l=1}^{L} R_{I(\mathbf{H}_l)} \overset{a.s}{=} \sum_{i=1}^{K} \Pr(I(\mathbf{H}_l)=i)R_i$
- Diversity gain: $P_e \doteq \mathrm{SNR}^{-d}$
- Outage Probability

$$P_{out,K} \triangleq \Pr(\log\det(\mathbf{I}_{N_r} + \mathbf{H}\mathbf{Q}_{I(\mathbf{H})}\mathbf{H}^{\mathrm{H}}) < R_{I(\mathbf{H})})$$

- Using a lower and upper bound, and taking SNR to inf., we can restrict our analysis to the case

$$\mathbf{Q}_i = \frac{P_i}{N_t}\mathbf{I}_{N_t} \quad \text{where } \mathrm{tr}(\mathbf{Q}_i) \le P_i$$

- Define:

$$I(\mathbf{H},\pi) \triangleq \log\det(\mathbf{I}_{N_r} + \frac{\pi}{N_t}\mathbf{H}\mathbf{H}^{\mathrm{H}})$$

$$F(\rho,\pi) \triangleq \Pr(I(\mathbf{H},\pi) < \rho)$$

Main Results – Single rate transmission

- Lemma 1: Outage minimizing power codebook $\{P_i^*\}_{i=1}^K$ solves the following optimization problem:

$$\max\ P_K\ \text{s.t}$$

$$[F(R,P_K)+1-F(R,P_1)]P_1+\sum_{i=2}^{K}[F(R,P_{i-1})-F(R,P_i)]P_i \le \text{SNR} \qquad 0 \le P_1 \le \ldots < P_K$$

- Then, the optimal index mapping is given by:

$$I^*(\mathbf{H}) = \begin{cases} 1 & \text{if } I(\mathbf{H},P_K^*) < R \\ \min\{i:\ i \in \{1,\ldots,K\},\ I(\mathbf{H},P_i^*) \ge R\} & \end{cases} \qquad 0 \le P_1 < \ldots < P_K$$

- Lemma 2: Using the Wishart's distribution of eigenvalues of matrix $\mathbf{HH}^{\mathrm{H}}$, after some mathematical manipulation we have:

$$F(r\log\text{SNR},\pi) \doteq \text{SNR}^{-D(r,p)} \quad \text{where,} \quad \pi \doteq \text{SNR}^{\text{p}}$$

$$D(r,p) \triangleq \inf_{\alpha_1^n \in \mathrm{A}} \sum_{1}^{n}(2i-1+m-n)\alpha_i, \quad \mathrm{A} \triangleq \{\alpha_1^n \mid \alpha_1 \ge \ldots \ge \alpha_n,\ \sum_{i=1}^{n}(p-\alpha_i)^+ < r\}$$

Main Results

- Theorem 1: The optimal D-M tradeoff of a single rate MIMO sytem with K quantization regions in the feedback link is upper bound by:

$$d^*_{out,K}(r) = D(r, 1 + d^*_{out,K-1}(r)) \quad \text{where} \quad d^*_{out,0}(r) \triangleq 0 \ \forall r$$

- When *r* is sufficiently close to 0 and $p \geq 1$,the minimization solution is achieved by choosing $\alpha_i^* = p$ for $i = 1, \ldots, n-1$ and $\alpha_n^* = p - r$,then

$\lim_{r \to 0} D(r, p) = N_t N_r p$, then by the theorem 1, we have

$$\lim_{r \to 0} d^*_{out,K}(r) = \sum_{i=1}^{K} (N_t N_r)^k$$

- It can also verify that $\lim_{r \to n} D(r, 1) = 0$, then $\lim_{r \to n} d^*_{out,K}(r) = 0, \ \forall \mathrm{K}$

- Similarly, in the case of adaptive rate transmission, we have:

$$d^*_{out,K}(r, r_{\min}) = D(r_{\min}, 1 + d^*_{out,K-1}(r, r_{\min})) \quad \text{where} \quad d^*_{out,1}(r, r_{\min}) \triangleq D(r, 1), \ \forall r \geq r_{\min}$$

$$\lim_{r_{\min} \to 0} d^*_{out,K}(r, r_{\min}) = (N_r N_t)^{K-1} D(r, 1) + \sum_{k=1}^{K} (N_t N_r)^k$$

Conditional Achievability of the optimal D-M Tradeoff

- Extended Approximately Universal Condition:

For any pair of codewords, let $\mu_1 \leq \ldots \leq \mu_n$ be n smallest squared singular values of $\Delta\mathbf{X}$, and

let $\mu_i = \mathrm{SNR}^{-\beta_i}$, then constrain the codebook $\mathbf{C}$ so that $(\min_{\mathbf{C}} \prod_{j=1}^{n} \mathrm{SNR}^{-(\beta_j)^+}) \dot{\geq} \mathrm{SNR}^{-r}$

- Consider the following index mapping

$$\underline{I}(\mathbf{H}) = \begin{cases} 1 & \text{if } I(\mathbf{H}, \underline{P}_K < (r+\varepsilon)\log \mathrm{SNR}, \\ \min\{i : i \in \{1,\ldots,K\}, & I(\mathbf{H}, \underline{P}_i) \geq (\mathrm{r}+\varepsilon)\log\mathrm{SNR} \end{cases}$$

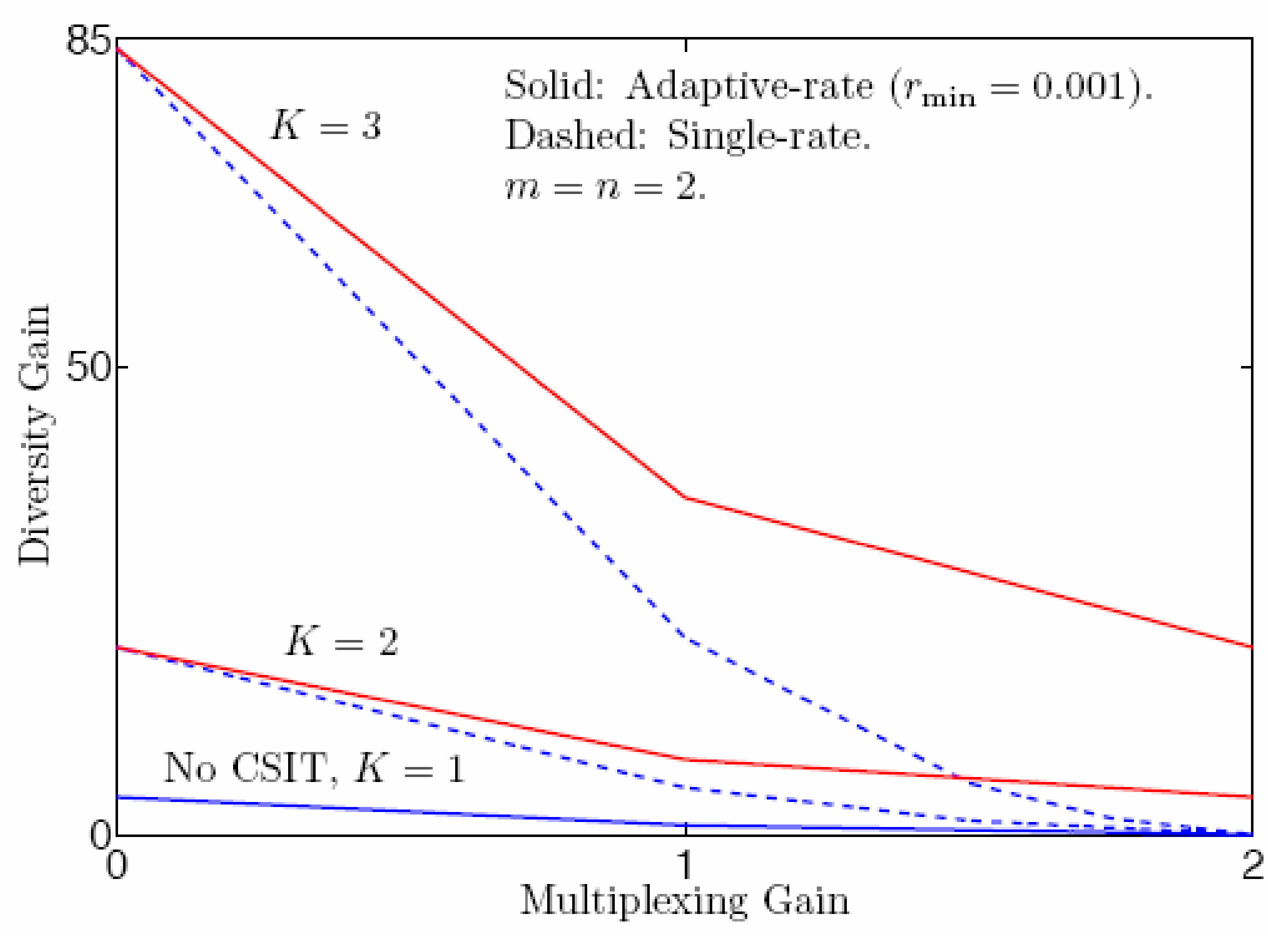
where $\underline{P}_1 = \dfrac{\mathrm{SNR}}{K}, \underline{P}_2 = \dfrac{\mathrm{SNR}}{KF((r+\varepsilon)\log \mathrm{SNR}, \underline{P}_1)}, \ldots, \underline{P}_K = \dfrac{\mathrm{SNR}}{KF((r+\varepsilon)\log \mathrm{SNR}, \underline{P}_{K-1})}$

then, construct the transmit codeword as $\mathbf{S}_i(k) = \sqrt{\dfrac{\underline{P}_i}{N_t}}\mathbf{X}(k), \quad i = 1,\ldots,K, \quad k = 1,\ldots,M$

Define the ith ε-outage-free region: $\overline{o}_i^{\varepsilon} = \{\mathbf{H} : \underline{I}(\mathbf{H}) = i, I(\mathbf{H}, P_i) \geq (r+\varepsilon)\log \mathrm{SNR}\}$

- Show the pairwise error probability in this region decay exponentially as SNR goes to inf., then the error probability is dominated by the outage event, which characterized by the D-M tradeoff

Numerical Example



Conclusions

- MIMO system provide both diversity and multiplexing gains, but there are tradeoff between them
- When there is no CSI at transmitter, the maximum diversity is $N_t N_r$
- With only a few bit feedback of quantized CSI, we can achieve much higher diversity gain, scaled exponentially with number of quantization regions.
- Especially, for the adaptive rate transmission, we can get non-zero diversity gain at maximum multiplexing gain with only a few bits of feedback

Thank You

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Questions?

References

- L. Zheng and D. Tse, “Diversity and Multiplexing: A Fundamental Tradeoff in Multiple Antenna Channels”, IEEE Transactions on Information Theory, vol. 49, May 2003, pp. 1073-96
- Tung T. Kim and Mikael Slogkund, “Diversity-Multiplexing Tradeoff in MIMO Channels with Partial CSIT”