that leading to (7), Theorem 1 implies

$$\begin{aligned}\frac{1}{n}\log\frac{1}{\epsilon_n} &\le \frac{1}{n}\log\frac{1}{1-\alpha}\\ &\quad + \frac{1}{n}\sup_{X^n}\log\left[1/P\left(\frac{1}{n}i_{X^nY^n}(X^n,Y^n) \le \frac{\log M}{n}+\frac{\log\alpha}{n}\right)\right]\\ &\le \frac{1}{n}\log\frac{1}{1-\alpha}\\ &\quad + \frac{1}{n}\sup_{X^n}\log\left[1/P\left(\frac{1}{n}i_{X^nY^n}(X^n,Y^n)\le R\right)\right]\end{aligned}$$

which holds for all sufficiently large $n$ because of (13). Then, (14) follows from (12) and the above inequality. □

We conjecture that the bound (14) is in fact tight; however, the known approaches to the constructive part of the coding theorem are not sufficient to prove this conjecture even for the simplest channels (for which the reliability function is not yet known for all rates). For example, in the case of a binary-symmetric channel, the evaluation of the right-hand side of (14) is an interesting unsolved large-deviations/optimization problem.

## III. Proof of the Bound

Theorem 1 admits a very simple proof that is quite different from the proofs of the special cases in [2] and [4].

*Proof:* Without loss of generality, we list the elements of $\mathcal{X}$ as the postive integers $1, 2, \cdots$. Let $Z_1, Z_2, \cdots,$ denote the random variables $\pi(1|Y), \pi(2|Y), \cdots$, placed in decreasing order, pointwise in the sample space[1] (it is immaterial how ties are resolved). First note from (3) that

$$\epsilon = 1 - E\{Z_1\}. \tag{15}$$

For any $\alpha \in [0,1]$, we can write

$$P(\pi(X|Y) > \alpha) = E\left\{\sum_{k\in\mathcal{X}} \pi(k|Y)1\{\pi(k|Y) > \alpha\}\right\} \tag{16}$$

where the expectation is with respect to the unconditional distribution of $Y$. The argument of the expected value in (16) can be written as

$$\sum_{k\in\mathcal{X}} \pi(k|Y)1\{\pi(k|Y) > \alpha\} = \sum_{k\in\mathcal{X}} Z_k 1\{Z_k > \alpha\}. \tag{17}$$

Dropping all but the first term

$$P(\pi(X|Y) > \alpha) \ge E\{Z_1 1\{Z_1 > \alpha\}\}. \tag{18}$$

In view of (15) and (18), all we need to do is to relate $E\{Z_1\}$ to $E\{Z_1 1\{Z_1 > \alpha\}\}$ using the fact that $0 \le Z_1 \le 1$. Since $Z_1 \le 1$ note that, for any $\alpha \in [0,1]$ we have

$$Z_1 = \alpha Z_1 + (1-\alpha)Z_1 \le \alpha + (1-\alpha)Z_1 1\{Z_1 > \alpha\} \tag{19}$$

which is tantamount to upperbounding $Z_1$ by $\alpha + (1-\alpha)Z_1$ when $\alpha \le Z_1 \le 1$, and by $\alpha$, otherwise.

Thus on combining (18) and (19), we have

$$\begin{aligned}E\{Z_1\} &\le \alpha + (1-\alpha)E\{Z_1 1\{Z_1 > \alpha\}\}\\ &\le \alpha + (1-\alpha)P(\pi(X|Y) > \alpha)\end{aligned} \tag{20}$$

which, together with (15), implies the bound. □

[1] That is, for each point $\omega$ in the underlying sample space, $Z_1(\omega)$, $Z_2(\omega), \cdots$, denotes the ordered sequence $\pi(1|Y(\omega)), \pi(1|Y(\omega)), \cdots$.

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# Asymptotic Efficiency of A Sequential Multihypothesis Test


Venugopal V. Veeravalli, *Member, IEEE*,
and Carl W. Baum, *Member, IEEE*



***Abstract*—A sequential multihypothesis test known as the MSPRT is generalized to account for nonuniform decision costs. Bounds on error probabilities and asymptotic expressions for the stopping time and error probabilities are given. A key result of this correspondence is a proof that the generalized MSPRT is asymptotically efficient.**



***Index Terms*— Sequential analysis, hypothesis testing, informational divergence.**


## I. Introduction

The sequential testing of more than two hypotheses has important applications in direct-sequence signal acquisition [1], [2], multiple-resolution-element radar [3], and other areas. Published work on sequential multihypothesis testing has generally taken two approaches. One approach has aimed at determining a Bayes optimal test, where optimality has been defined in terms of the minimization of a linear combination of two quantities: the expected decision cost and the expected number of observations taken by the test. A recursive solution to the Bayesian optimization problem has in fact been obtained [4]–[6], but unfortunately, this solution is very complex and impractical except in a few special cases.

A second approach has focused on extending and generalizing the sequential probability ratio test (SPRT), a binary test, to incorporate more than two hypotheses. A survey of many of these tests is found in [7]. Although these tests are of low complexity, they have been developed without much consideration to optimality.

In [8], a test is given that incorporates both approaches. The test, called the $M$-ary Sequential Probability Ratio Test (MSPRT), is a generalization of the SPRT. The MSPRT has a simple structure that facilitates implementation, and it is also based on the solution to the Bayesian optimization problem. It is shown in [8] that the MSPRT approximates the Bayes optimal test, and an example demonstrates that, in at least some cases, the MSPRT is asymptotically optimal as the cost per observation decreases to zero. The MSPRT test structure


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V. V. Veeravalli is with the Department of Electrical and Computer Engineering, Rice University, Houston, TX 77251-1892 USA.

C. W. Baum is with the Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29634-0915 USA.



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has also been shown to be asymptotically optimal as the distance between hypotheses becomes infinite [4].

In this correspondence, the MSPRT is first generalized to incorporate nonuniform decision costs. The bounds and asymptotic expressions for the error probabilities and expected stopping times given in [8] are generalized correspondingly. It is then shown that the generalized MSPRT is in fact *asymptotically efficient.*

## II. A Generalized MSPRT

Let $X_1, X_2, \cdots$ be an infinite sequence of observations, independent and identically distributed with density $f$, and let $H_j$ be the hypothesis that $f = f_j$ for $j = 0, 1, \cdots, M-1$. Assume that the Kullback–Leibler distance between $f_k$ and $f_j$ is positive and finite for all $j \neq k$. Based on the observations, a decision must be made as to which hypothesis is true. Let $W(j,k)$ denote the cost of deciding $H_k$ when $H_j$ is true, and assume that $W(j,k) > 0$ for $j \neq k$ and $W(k,k) = 0$. Let $\alpha_{j,k}$ denote the probability of deciding $H_k$, conditioned on $H_j$ being the true hypothesis. Let $r_k$ denote the expected cost (risk) associated with deciding $H_k$. Clearly,

$$r_k = \sum_{j:j\neq k} \pi_j W(j,k) \alpha_{j,k} \tag{1}$$

where $\pi_j$ is the *a priori* probability of $H_j$. The total expected decision cost $R$ is the sum $r_0 + r_1 + \cdots + r_{M-1}$.

We consider a generalization of the $M$-ary Sequential Probability Ratio Test (MSPRT) [8] for this decision-making problem. Specifically, the stopping time $N_A$ and the final decision $\delta$ are defined as

$$N_A = \text{first } n \geq 1 \text{ such that, for at least one } k$$

$$\pi_k A_k \prod_{i=1}^{n} f_k(X_i) > \sum_{j:j\neq k} \pi_j W(j,k) \prod_{i=1}^{n} f_j(X_i)$$

$$\delta = H_m, \quad \text{where } m = \arg\min_k \sum_{j:j\neq k} \pi_j W(j,k) \prod_{i=1}^{N_A} f_j(X_i)$$

where the parameters $\{A_k\}$ are all taken to be positive. The MSPRT is simply this test restricted to uniform costs, i.e., $W(j,k) = 1$ for $j \neq k$. In this correspondence, we abuse terminology slightly and use the term MSPRT to denote the more general test given above.

The case when the parameters $\{A_k\}$ are small is usually of primary interest because it corresponds to small risk values $\{r_k\}$. If, for each $k$

$$A_k \leq \min_{j,\ell:j\neq \ell} W(j,\ell)$$

then it is easy to show that the inequality in the definition of $N_A$ is satisfied by *at most* one value of $k$. In this case, the MSPRT takes the simpler form

$$N_A = \text{first } n \geq 1 \text{ such that, for some } k$$

$$\pi_k A_k \prod_{i=1}^{n} f_k(X_i) > \sum_{j:j\neq k} \pi_j W(j,k) \prod_{i=1}^{n} f_j(X_i)$$

$$\delta = H_k.$$

We have generalized a number of results regarding the MSPRT in [8] that relate to bounds and characterizations of the asymptotic performance. The proofs of these results are straightforward extensions of the proofs in [8], and are omitted.

*Theorem 1:* The stopping time $N_A$ is exponentially bounded (and, therefore, finite with probability one), conditioned on each of the hypotheses $H_k$, $k = 0, \cdots, M-1$.

*Theorem 2:* For each $k$, $r_k \leq \pi_k A_k$.

The following are asymptotic results under the condition $\max_\ell A_\ell \to 0$. (Note that $D(f,g)$ denotes the Kullback–Leibler distance between densities $f$ and $g$.)

*Lemma 1:* For each $k$, $N_A \to \infty$ a.s.-$f_k$ as $\max_\ell A_\ell \to 0$.

*Theorem 3:* As $\max_\ell A_\ell \to 0$

$$\frac{N_A}{-\log A_k} \to \frac{1}{\min_{j:j\neq k} D(f_k, f_j)} \text{ a.s.-} f_k$$

and

$$\frac{\mathrm{E}_{f_k}[N_A]}{-\log A_k} \to \frac{1}{\min_{j:j\neq k} D(f_k, f_j)}.$$

*Theorem 4:* Assume that the density $f_{j^*}$ that minimizes $D(f_k, f_j)$ is unique, and assume that the log-likelihood ratios are nonarithmetic. Then

$$\frac{r_k}{\pi_k A_k} \to \gamma_k$$

as $\max_\ell A_\ell \to 0$, where $\gamma_k = \mathrm{E}_{f_k}[e^{-W_k}]$, and $W_k$ has distribution

$$\mathrm{P}_{f_k}(W_k \leq w) = \frac{\int_0^w \mathrm{P}_{f_k}\left(\sum_{i=1}^{\tau_k^+} \log \frac{f_k(X_i)}{f_{j^*}(X_i)} > s\right) ds}{\mathrm{E}_{f_k}\left[\sum_{i=1}^{\tau_k^+} \log \frac{f_k(X_i)}{f_{j^*}(X_i)}\right]}$$

and $\tau_k^+$ is the first $n \geq 1$ such that

$$\sum_{i=1}^{n} \log \frac{f_k(X_i)}{f_{j^*}(X_i)} > 0.$$

It is easily shown that $0 < \gamma_k < 1$ for each $k$. See [9] for a presentation of techniques for evaluating $\gamma_k$.

## III. Asymptotic Efficiency

We begin this section by discussing the Bayes optimal multihypothesis sequential test. Let the true hypothesis be denoted by a random variable $H$. Note that $H$ takes the value $j$ with probability $\pi_j$ for $j = 0, \cdots, M-1$. Consider a sequential test with stopping time $N$ and final decision $\delta$. The expected decision cost (total risk) of this test is $\mathrm{E}[W(\delta, H)]$. The Bayesian optimization problem is to find a sequential test that minimizes the linear combination $\mathrm{E}[cN + W(\delta, H)]$.

The parameter $c$ may be interpreted as the cost per observation.

For uniform decision costs, it is shown in [8] that the MSPRT approximates the much more complicated optimal test when $c$ approaches zero. There are indications that the MSPRT may be asymptotically optimal as $c \to 0$ (an example for which the MSPRT is indeed asymptotically optimal is given in [8]). Although we have not been able to establish such an asymptotic optimality result, we show in this correspondence that the MSPRT is asymptotically *efficient.* Toward this end, we first prove the following lemma:

*Lemma 2:* Consider any $M$-ary test procedure with finite (a.s.) stopping time $N$ and error probabilities $\{\alpha'_{j,k}\}$. Then the following inequality holds for each $j \neq k$:

$$\mathrm{E}_{f_k}[N] \geq \frac{1}{D(f_k, f_j)} \sum_{\ell=0}^{M-1} \alpha'_{k,\ell} \log \frac{\alpha'_{k,\ell}}{\alpha'_{j,\ell}}.$$

$$\begin{aligned}
\mathrm{E}_{f_k}\left[\sum_{i=1}^{N} \log \frac{f_k(X_i)}{f_j(X_i)}\right] &= \sum_{\ell=0}^{M-1} \alpha'_{k,\ell}\, \mathrm{E}_{f_k}\left[\sum_{i=1}^{N} \log \frac{f_k(X_i)}{f_j(X_i)} \,\middle|\, \text{accept } H_\ell\right] \\
&\geq -\sum_{\ell=0}^{M-1} \alpha'_{k,\ell} \log\left(\mathrm{E}_{f_k}\left[\prod_{i=1}^{N} \frac{f_j(X_i)}{f_k(X_i)} \,\middle|\, \text{accept } H_\ell\right]\right) \\
&= -\sum_{\ell=0}^{M-1} \alpha'_{k,\ell} \log\left(\frac{1}{\alpha'_{k,\ell}} \mathrm{E}_{f_k}\left[\prod_{i=1}^{N} \frac{f_j(X_i)}{f_k(X_i)} 1_{\{\text{accept} H_\ell\}}\right]\right) \\
&= -\sum_{\ell=0}^{M-1} \alpha'_{k,\ell} \log \frac{\alpha'_{j,\ell}}{\alpha'_{k,\ell}} = \sum_{\ell=0}^{M-1} \alpha'_{k,\ell} \log \frac{\alpha'_{k,\ell}}{\alpha'_{j,\ell}}
\end{aligned} \tag{3}$$

*Proof:* By the Wald Identity [10], we have

$$\mathrm{E}_{f_k}[N] = \frac{1}{D(f_k, f_j)} \mathrm{E}_{f_k}\left[\sum_{i=1}^{N} \log \frac{f_k(X_i)}{f_j(X_i)}\right]. \tag{2}$$

Now (see (3) at the top of this page) where the second line is due to Jensen's inequality and $1_{\{\cdot\}}$ is the indicator function.

The lemma follows from (2) and (3).

Using this result, we can show that the MSPRT is asymptotically efficient:

*Theorem 5:* Consider an MSPRT with parameters $\{A_k\}$, stopping time $N_A$, and corresponding risk values $\{r_k\}$. Consider any other test procedure with finite (a.s.) stopping time $N$ and risk values $r'_k \leq r_k$ for each $k$. Then, for each $k$

$$\liminf_{\max_\ell A_\ell \to 0} \frac{\mathrm{E}_{f_k}[N]}{\mathrm{E}_{f_k}[N_A]} \geq 1.$$

*Proof:* Let $\{\alpha_{j,k}\}$ and $\{\alpha'_{j,k}\}$ denote, respectively, the error probabilities of the MSPRT and the other test procedure. Since

$$\sum_{\ell=0}^{M-1} \alpha'_{k,\ell} = 1$$

it follows that

$$-\sum_{\ell=0}^{M-1} \alpha'_{k,\ell} \log \alpha'_{k,\ell} \geq -\log M. \tag{4}$$

Applying Lemma 2 and (4), it follows that, for each $j \neq k$

$$\mathrm{E}_{f_k}[N] \geq \frac{-\log M - \alpha'_{k,k} \log \alpha'_{j,k}}{\min_{\ell \neq k} D(f_k, f_\ell)}. \tag{5}$$

Now, an application of Theorem 2 gives us the following bounds:

$$\alpha'_{j,k} \leq \frac{r'_k}{\pi_j W(j,k)} \leq \frac{r_k}{\pi_j W(j,k)} \leq \frac{\pi_k A_k}{\pi_j W(j,k)}$$

and

$$\begin{aligned}
\alpha'_{k,k} &= 1 - \sum_{\ell \neq k} \alpha_{k,\ell} \geq 1 - \sum_{\ell \neq k} \frac{r'_\ell}{\pi_k W(k,\ell)} \\
&\geq 1 - \sum_{\ell \neq k} \frac{r_\ell}{\pi_k W(k,\ell)} \geq 1 - \sum_{\ell \neq k} \frac{\pi_\ell A_\ell}{\pi_k W(k,\ell)}.
\end{aligned}$$

Thus for each $j \neq k$

$$\mathrm{E}_{f_k}[N] \geq \frac{-\log M + \left(1 - \sum_{\ell \neq k} \frac{\pi_\ell A_\ell}{\pi_k W(k,\ell)}\right) \log \frac{\pi_j W(j,k)}{\pi_k A_k}}{\min_{\ell \neq k} D(f_k, f_\ell)}.$$

Pick any $j \neq k$. Then, from the previous inequality we obtain

$$\begin{aligned}
\frac{\mathrm{E}_{f_k}[N]}{\mathrm{E}_{f_k}[N_A]} \geq{}& \frac{-\log M + \left(1 - \sum_{\ell \neq k} \frac{\pi_\ell A_\ell}{\pi_k W(k,\ell)}\right) \log \frac{\pi_j W(j,k)}{\pi_k}}{\mathrm{E}_{f_k}[N_A] \min_{\ell \neq k} D(f_k, f_\ell)} \\
&+ \frac{-\log A_k \left(1 - \sum_{\ell \neq k} \frac{\pi_\ell A_\ell}{\pi_k W(k,\ell)}\right)}{\mathrm{E}_{f_k}[N_A] \min_{\ell \neq k} D(f_k, f_\ell)}.
\end{aligned} \tag{6}$$

The first term on the right—hand side of (6) goes to 0 by Lemma 1 and the second term goes to 1 by Theorem 3, both as $\max_\ell A_\ell \to 0$. The result follows.

The result in Theorem 5 is given in terms of asymptotics as the MSPRT parameters go to zero. The asymptotic efficiency result for the case when the risk values go to zero is presented in the following corollary to Theorem 5. The only additional condition required for the corollary is that the densities $\{f_k\}$ have identical support; this condition is implied by the assumption made in Section II that the Kullback–Leibler distance between $f_k$ and $f_j$ is finite for all $j \neq k$.

*Corollary 1:* Consider an MSPRT with parameters $\{A_k\}$, stopping time $N_A$, and corresponding risk values $\{r_k\}$. Consider any other test procedure with finite (a.s.) stopping time $N$ and risk values $r'_k \leq r_k$ for each $k$. Then

$$\liminf_{\max_\ell r_\ell \to 0} \frac{\mathrm{E}_{f_k}[N]}{\mathrm{E}_{f_k}[N_A]} \geq 1.$$

*Proof:* To prove this result, we simply need to establish that $\max_\ell r_\ell \to 0$ implies that $\max_\ell A_\ell \to 0$.

Since the densities $\{f_k\}$ have identical support, any set $S$ consisting of observation sequences of finite length that satisfies $P_{f_k}(S) = 0$ for some $k$ must satisfy the condition $P_{f_k}(S) = 0$ for all $k$.

Now, suppose $A_k \geq a > 0$ and $r_k = 0$. Let $\Delta_k$ denote the decision region for $H_k$, i.e., $\Delta_k$ consists of all observation sequences of finite length (since $N_A$ is finite with probability one) that result in a choice of $H_k$. Since $r_k = 0$, we must have $P_{f_j}(\Delta_k) = 0$ for all $j \neq k$, which implies that $P_{f_k}(\Delta_k) = 0$ as well. Thus $H_k$ is never chosen. But if $H_k$ is never chosen, then the total risk

$$R \geq \pi_k \min_{j,\ell : j \neq \ell} W(j,\ell) > 0.$$

But by letting each of $\{A_j, j \neq k\}$ go to zero we should be able to make $R$ as small as we like by Theorem 2. This is a contradiction. Thus $r_k$ cannot equal 0 if $A_k \geq a > 0$. This means that if $A_k$ does not converge to zero, $r_k$ does not converge to zero either. The corollary follows.

## IV. Discussion

As already noted, the MSPRT is a generalization of the SPRT to more than two hypotheses. In the binary case, it is well known that the SPRT is not just asymptotically efficient, it is in fact optimal; i.e., given a set of risks (or error probabilities), the conditional expected stopping time under each hypothesis is minimized by the SPRT [11].

In the case of more than two hypotheses, the Bayes optimal test is known, but its complex structure makes implementation very impractical. In contrast, the MSPRT is an intuitively appealing test which can be expressed in terms of simple combinations of likelihood ratios. Thus in conjunction with the result of Theorem 5 and its corollary, we conclude that the MSPRT is a practical choice for multihypothesis testing and is especially recommended for applications in which risk requirements are stringent.

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# On the Performance Degradation from One-Bit Quantized Detection

Peter Willett, *Member, IEEE*, and Peter F. Swaszek, *Member, IEEE*

***Abstract*— It is common signal detection practice to base tests on *quantized* data and frequently, as in decentralized detection, this quantization is extreme: to a single bit. As to the accompanying degradation in performance, certain cases (such as that of an additive signal model and an efficacy measure) are well-understood. However, there has been little treatment of more general cases. In this correspondence we explore the possible performance loss from two perspectives. We examine the Chernoff exponent and discover a nontrivial lower bound on the relative efficiency of an optimized one-bit quantized detector as compared to unquantized. We then examine the case of finite sample size and discover a family of nontrivial bounds. These are upper bounds on the probability of detection for an unquantized system given a specified quantized performance, given that both systems operate at the same false-alarm rate.**

***Index Terms*—Decentralized detection, quantized detection, sign detector, Chernoff bounds.**

## I. Introduction

In many detection problems it is necessary to quantize data prior to decision-making, and naturally this quantization operation can degrade performance. In particular, for *decentralized* detection, where the decision-making operation is usually known as fusion, the quantization is often to two levels only. It is reasonable, therefore, to be concerned with the effect of such quantization, and to wish to make statements as to the maximal loss attributable thereto. That is the aim of this correspondence.

To begin with, consider the classical problem of the detection of a known signal $\{s_i\}$ in independent and identically distributed (i.i.d.) additive noise $\{n_i\}$

$$\begin{aligned} &H_0: x_i = n_i \\ &H_1: x_i = n_i + \theta s_i \qquad i = 1, 2, \cdots, N \end{aligned} \tag{1}$$

($\theta$ is a multiplicative constant), and the equally classical *sign-correlator*

$$\gamma_s(\mathbf{x}) = \sum_{i=1}^{N} s_i \operatorname{sgn}(x_i) \underset{H_0}{\overset{H_1}{\gtrless}} \tau. \tag{2}$$

In this case, our fundamental question is as to the possible performance degradation from the use of the sign-correlator, as compared to the optimal (i.e., likelihood ratio) test statistic.

A partial answer is available from asymptotics; that is, in the case that $N \to \infty$ and $\theta$ decreases to zero as $1/\sqrt{N}$. Here the measure of interest is *efficacy* (see, for example, [1])

$$\eta(\gamma) = \lim_{N\to\infty} \frac{1}{N} \frac{\left[\frac{\partial}{\partial\theta} E_\theta\{\gamma\}|_{\theta=0}\right]^2}{\text{Var}_0\{\gamma\}} \tag{3}$$

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P. Willett is with the University of Connecticut, Storrs, CT 06269 USA.
P. F. Swaszek is with the University of Rhode Island, Kingston, RI 02881 USA.

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