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MMSE Detection in Asynchronous CDMA Systems: An Equivalence Result

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***Abstract*—The analysis of linear minimum mean-square error (MMSE) detection in a band-limited code-division multiple-access (CDMA) system that employs random spreading sequences is considered. The key features of the analysis are that the users are allowed to be completely asynchronous, and that the chip waveform is assumed to be the ideal Nyquist sinc function. It is shown that the asymptotic signal-to-interference ratio (SIR) at the detector output is the same as that in an equivalent chip-synchronous system. It is hence been established that synchronous analyses of linear MMSE detection can provide useful guidelines for the performance in asynchronous band-limited systems.**

***Index Terms*—Asymptotic analysis, asynchronous systems, band-limited communication, code-division multiple access (CDMA), least mean squares methods, matched filters (MFs), minimum mean-square error (MMSE) detection, sinc function.**

I. Introduction

Multiuser detection in code-division multiple-access (CDMA) systems has been a topic of intense research for more than a decade [1]. Several criteria have been used for designing multiuser detectors, and a particularly appealing one is to minimize the mean-squared error (MSE) of the symbol estimates at the output of the detector. When the detector is further constrained to be linear we obtain the linear minimum mean-squared error (LMMSE or simply, MMSE) detector [2]. Equivalently, the MMSE detector also maximizes the output signal-to-interference ratio (SIR) over the class of linear detectors. In addition, it allows for an adaptive implementation [3]. Hence, the MMSE detector has been a subject of considerable study.

Detailed performance analysis for the MMSE detector was first considered in [4]. The spreading sequences were assumed to be arbitrary but fixed, and the Gaussianity of the multiaccess interference at the output of the detector was analyzed under various asymptotic scenarios. A more promising approach for analysis was introduced in [5], [6]. Here, the spreading sequences were treated as independent random vectors, and limits of the SIR and capacity were studied as the number of users (K) and the processing gain (N) tend to infinity with the ratio K/N approaching a constant. The limitation of the analysis in [5], [6] is that it is restricted to the situation where the users are symbol-synchronous. In [7], the SIR analysis of [5] was extended to the case where the users are symbol-asynchronous but chip-synchronous, i.e., the delays of all the users are aligned to the chip timing.

While it allows for accurate large-system analysis, the synchronous or chip-synchronous assumption is not realistic for the received signal on the reverse link of a cellular CDMA system, especially with user mobility and the resulting variations in the delay. Thus, we would like to allow the users to be completely asynchronous, i.e., symbol- as well as chip-asynchronous. Analysis of the MMSE detector with random spreading sequences and completely asynchronous users was considered in [8]. However, the performance measure was the average near–far resistance of the detector and bounds were obtained on this quantity for finite K and N. Furthermore, the analysis relied on the assumption that the chip waveform was limited to a chip interval.

In this correspondence, we allow the users to be completely asynchronous and consider SIR at the detector output as the performance metric. We also assume that the system employs the ideal band-limited (and hence, of infinite duration) sinc chip waveform. For single-user narrow-band systems, the sinc waveform maximizes the signaling rate when the symbol waveforms are constrained to have a given bandwidth and to have no intersymbol interference [9]. In spread-spectrum systems, we have an additional degree of freedom, since the processing gain of the system can be varied with the excess bandwidth of the chip waveform to keep the symbol rate and occupied bandwidth fixed. In such a framework, the sinc waveform maximizes the processing gain since it has zero excess bandwidth. For the matched-filter (MF) detector, the maximum processing gain also results in the maximum output SIR across all waveforms [10], [11]. Hence, practical CDMA systems (e.g., [12]) employ waveforms that have an approximately flat spectrum over the band of operation. Similar observations hold for the MMSE detector as well, although a formal proof of the optimality of the sinc waveform appears to be open [13]. Based on the above remarks, the sinc waveform can be considered to be a benchmark for band-limited systems. Hence, analysis of the MMSE detector when the users are completely asynchronous and employ the sinc waveform is of much interest, from a theoretical as well as a practical viewpoint.

II. System Model and MF Detection

We consider a direct-sequence CDMA (DS/CDMA) model with $K+1$ users, where the received complex baseband signal is given by

$$r(t) = \sum_{k=0}^{K} s_k(t - \tau_k T_c) e^{i\phi_k} + w(t), \qquad t \in [-\infty, \infty] \tag{1}$$

where $s_k(t)$ is the signal transmitted by user k

$$s_k(t) = \sum_{m=-\infty}^{\infty} \sqrt{\mathcal{E}_k}\, b_k^{(m)} c_k^{(m)}(t). \tag{2}$$

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The notation used in (1) and (2) is as follows. The quantity $b_k^{(m)}$ is symbol m of user k, and

$$c_k^{(m)}(t) = \sum_{n=0}^{N-1} c_{k,n}^{(m)}\,\psi(t - mT_s - nT_c)$$

is its spreading waveform. Here T_s and T_c are the symbol and chip periods, respectively, and $N = T_s/T_c$ is the processing gain of the system. As discussed in Section I, the results of this correspondence are derived for the case where $\psi(t)$ is the sinc chip waveform (normalized to have unit energy). To distinguish between statements that are applicable to a general chip waveform and those that hold only for the sinc pulse, we denote the specific sinc waveform by $\psi^\star(t)$

$$\psi^\star(t) = \frac{1}{\sqrt{T_c}}\,\mathrm{sinc}\left(\frac{t}{T_c}\right)$$

where

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}.$$

Furthermore, in (1) and (2), $\phi_k,\ \tau_k$ and $\mathcal{E}_k$ are the carrier phase offset, delay, and symbol energy of user k, respectively. Finally, $w(t)$ is a zero mean proper complex Gaussian process with two-sided power spectral density N_0, i.e.,

$$R_w(\tau) = E[w^*(t)w(t+\tau)] = N_0\delta(\tau).$$

Since the sinc function is of infinite duration, we have allowed the observation interval for the continuous time CDMA signal to be infinite. In addition, we make the following assumptions throughout this correspondence.

- The delays τ_k are normalized to the chip period T_c and take on real values in $[0,\ N]$. When τ_k is restricted to be an integer, the users are *chip*-synchronous. In particular, when $\tau_k = 0\ \forall k$, the users are *symbol*-synchronous.
- The desired user corresponds to $k = 0$, and the timing reference at the receiver is synchronized to the desired user, so that $\tau_0 = 0$.
- The chips $c_{k,n}^{(m)}$ are modeled as complex, *independent, and identically distributed (i.i.d.)*, variance $1/N$ random variables, with finite fourth moments. In addition, the symbols are modeled as i.i.d. zero mean, unit variance random variables.[1]

We begin with a review of the analysis for the conventional MF detector [10]. The desired symbol of user 0 is taken to be $b_0^{(0)}$. The MF statistic is then obtained through correlation with the corresponding spreading waveform

$$X_{\mathrm{MF}} = \int_{-\infty}^{\infty} r(t)c_0^{(0)}(t)e^{-i\phi_0}\,dt. \tag{3}$$

The performance metric used is the SIR (Γ) at the output of the detector. With the MF, the SIR for $b_0^{(0)}$ is defined as

$$\Gamma_0^{\mathrm{MF}} = \frac{\mathrm{E}_{b_0}\,|\mathrm{E}[X_{\mathrm{MF}}|b_0]|^2}{\mathrm{Var}[X_{\mathrm{MF}}|b_0]} \tag{4}$$

where the expectation is taken over the sequences of all the users, and the symbols and delays of the interferers. If the delays $\{\tau_k\}$ are modeled to be uniform in $[0,\ N]$, then, for a general chip waveform $\psi(t)$ (see [10])

$$\Gamma_0^{\mathrm{MF}} = \frac{\mathcal{E}_1}{N_0 + \frac{\sigma_\psi}{N}\sum_{k=1}^{K}\mathcal{E}_k} \tag{5}$$

where

$$\sigma_\psi = \frac{1}{T_c}\int_{-\infty}^{\infty}|\Psi(f)|^4\,df$$

with $\Psi(f)$ being the Fourier transform of $\psi(t)$. It is also shown in [10] that, if $\Psi(f)$ is limited to a bandwidth W, the sinc waveform $\psi^\star(t)$ with $T_c = \frac{1}{2W}$ minimizes the quantity $\int_{-\infty}^{\infty}|\Psi(f)|^4\,df$. Thus, under equal bandwidth and symbol rate constraints, $\psi^\star(t)$ maximizes the output SIR of the MF detector (see also [11]). In addition, when $\psi(t) = \psi^\star(t)$, we have $\sigma_\psi = 1$, and

$$\Gamma_0^{\mathrm{MF}} = \frac{\mathcal{E}_1}{N_0 + \frac{1}{N}\sum_{k=2}^{K}\mathcal{E}_k}.$$

It can be easily seen that the above SIR is the same as that obtained in a symbol-synchronous system (i.e., $\tau_k = 0\ \forall k$), with K users, processing gain N, and i.i.d. random spreading sequences. We refer to this equality as the equivalence result for the MF detector. We will be interested in establishing a similar equivalence result for MMSE detection in the remainder of the correspondence.[2]

For this purpose, it is of interest to note that the equivalence for the MF detector holds even when we do not average over the delays of the asynchronous interferers. With τ_k fixed, the variance of the interference in the asynchronous case takes on the form

$$\frac{1}{N}\sum_{k=2}^{K}\mathcal{E}_k\sum_{j=-\infty}^{\infty}\mathrm{sinc}^2(j+\tau_k)$$

and the equivalence follows immediately from the following key property of the sinc waveform:

$$\sum_{j=-\infty}^{\infty}\mathrm{sinc}^2(j+\tau) = 1, \qquad \forall\tau. \tag{6}$$

We also note that the above equivalence is obtained for a finite system, with SIR in (5) defined through an average over the spreading sequences. Alternately, we can obtain the equivalence without averaging over the sequences or the delays, but under the large-system asymptote of $K,\ N \to \infty$ with $K/N \to \beta$.

Result 1: Under the random sequence model, the SIR of the MF detector converges in mean square to that in the symbol synchronous case as $K,\ N \to \infty$ with $K/N \to \beta$. The limiting SIR is

$$\Gamma_0^{\mathrm{MF}} = \frac{\mathcal{E}_1}{N_0 + \beta\mathrm{E}_{\mathcal{E}}\,\mathcal{E}}$$

where the expectation is over the limiting empirical distribution of the symbol energies $\{\mathcal{E}_k\}$, and this distribution is assumed to exist.

The result can be proved in a straightforward manner using techniques similar to those used in [7], along with the property (6). We now consider the equivalence result for linear MMSE detection.

III. MMSE Detector: Problem Formulation

In formulating the SIR problem for MMSE detection, we need to consider a few additional issues and make appropriate assumptions. While it is possible to derive the MMSE detector with an infinite se-

[1]Note that the independence of sequences across symbol index m amounts to assuming long spreading sequences. The analysis of this correspondence could be extended to a short sequence system where different symbols of a given user employ the same spreading sequence, but we make the long sequence assumption for the sake of simplicity.

[2]It can also be seen that the SIR in the symbol and chip-synchronous cases are equal for the MF detector. However, the distinction between these two cases will be important for the MMSE detector.

quence of symbols transmitted by each user, the analysis appears difficult. Hence, we assume that the desired user transmits only M symbols, indexed from $m = 0$ to $m = M - 1$. Note that, under the ideal sinc waveform assumption, each symbol occupies an infinite time duration. However, with delay $\tau_0 = 0$, we can think of each symbol $b_0^{(m)}$ as corresponding to the interval $[mT_s,\ (m+1)T_s]$. Furthermore, we assume that $M = 2p + 1$, and the symbol of interest is taken to be the $b_0^{(p)}$, which "occurs" at the center of the interval $[0,\ MT_s]$.

Since the interferers are asynchronous, we assume that $M + 1$ symbols are transmitted by each interferer, with an additional symbol[3] "occurring" at the left of the interval $[0,\ MT_s]$. The interfering symbols of user k are indexed from $m = -1$ to $m = M - 1$. Hence, the analysis can be thought of as corresponding to a multishot detector over an M-symbol observation.

For convenience in notation, we now reindex the symbols in (1) by using a single index $j = k(M + 1) + m$. Since there are a total of $K_e = M + K(M + 1)$ symbols, we have

$$r(t) = \sum_{j=0}^{K_e-1} A_j b_j c_j(t - \tau_j' T_c) e^{i\phi_j'} + w(t)$$

where, for $j = k(M + 1) + m$, $b_j = b_k^{(m)}$, $c_j(t) = c_k^{(m)}(t)$, $A_j = \sqrt{\mathcal{E}_k}$, $\phi_j' = \phi_k$, and $\tau_j' = \tau_k + mN$. For further simplification, we abuse notation slightly and drop the primes in τ_j' to have

$$r(t) = \sum_{j=0}^{K_e-1} A_j b_j c_j(t - \tau_j T_c) e^{i\phi_j} + w(t). \tag{7}$$

Thus, we think of j as indexing K_e effective users, with the implicit understanding that across the symbols of the same actual user, the amplitudes A_j are equal and the delays τ_j are related through linear shifts. With the above reindexing, the desired symbol becomes b_p and the first interfering symbol becomes b_M.

The MMSE detector for b_p is more conveniently expressed and analyzed in the discrete-time domain. It is possible to generate K_e discrete sufficient statistics by correlating with the spreading waveforms of each symbol transmitted by each user. These statistics are sufficient for joint detection of all the symbols of all the users. We can then derive the linear MMSE detector based on these correlation statistics. The correlation approach was used to analyze the MMSE detector in [4], [6] for the symbol-synchronous case. However, for the completely asynchronous case, analysis with this approach again appears difficult. Instead, we assume that MN statistics are generated by sampling the output of a chip-MF once every chip interval

$$\begin{aligned} y_n &= \int_{-\infty}^{\infty} r(t)\psi(t - nT_c)\, dt, \qquad n = 0,\ \ldots,\ MN - 1 \\ \boldsymbol{y} &:= [y_0,\ y_1,\ \ldots,\ y_{MN-1}]^\top. \end{aligned} \tag{8}$$

This approach to obtaining the discrete system model is followed in the MMSE analysis in [5], [7]. Note that the statistics generated are sufficient only under the assumption of synchronous and chip-synchronous users, and are not sufficient in the general asynchronous case [13]. In particular, with the sinc waveform assumption, while the above sampling rate is equal to the Nyquist rate, the loss in sufficiency is due to the fact that we have restricted ourselves to a *finite* number of statistics. However, we expect the loss in sufficiency to go to zero as $M \to \infty$, since the sinc functions would then span the received signal. In the analysis for finite M below, we derive the MMSE detector for b_p based on the observation $\boldsymbol{y}$ in (8), and consider any loss in sufficiency to be a part of the suboptimality of the detector.

[3]We could have included this additional symbol for the desired user as well. But we choose to ignore this symbol since it simplifies the notation and does not affect the analysis.

Now, since chip-matched filtering is a linear operation, the discrete system model is additive across the transmitted symbols, and we have

$$\boldsymbol{y} = \sum_{j=0}^{K_e-1} b_j \boldsymbol{s}_j + \boldsymbol{w} \tag{9}$$

where $\boldsymbol{w}$ is a zero-mean white Gaussian vector with variance $\sigma^2 = N_0$, and $\boldsymbol{s}_j$ is a vector of length MN with components

$$\boldsymbol{s}_j(n) = \int_{-\infty}^{\infty} A_j c_j(t - \tau_j T_c)\psi(t - nT_c)\, dt, \quad n = 0, \ldots, MN - 1.$$

This implies that

$$\boldsymbol{s}_j = A_j e^{i\phi_j} \boldsymbol{R}_\psi(\tau_j) \boldsymbol{c}_j \tag{10}$$

where $\boldsymbol{c}_j$ is the i.i.d. spreading sequence of effective user j, and

$$\begin{aligned} \boldsymbol{R}_\psi(\tau_j)[n,\ \ell] &= R_\psi(\tau_j + \ell - n), \\ & n = 0,\ \ldots,\ MN - 1;\ \ell = 0,\ \ldots,\ N - 1. \end{aligned}$$

Here

$$R_\psi(\tau) = \int_{-\infty}^{\infty} \psi(t)\psi(t - \tau)\, dt$$

is the autocorrelation function of the chip waveform. In general, $\boldsymbol{R}(\tau_j)$ is an $MN \times N$ Toeplitz matrix that involves only the correlation function R_ψ and the delay τ_j. For the sinc waveform, $R_{\psi^\star}(\tau) = \mathrm{sinc}(\tau)$. For brevity in notation, we denote the matrix $\boldsymbol{R}_{\psi^\star}(\tau_j)$ by $\boldsymbol{R}(\tau_j)$, so that

$$\boldsymbol{R}(\tau_j)[n,\ \ell] = \mathrm{sinc}(\tau_j + \ell - n). \tag{11}$$

Based on the observation $\boldsymbol{y}$ in (9), the linear MMSE estimate for b_p is given by [2]

$$\hat{b}_p = \frac{\boldsymbol{s}_p^\dagger \boldsymbol{B}_p \boldsymbol{y}}{1 + \boldsymbol{s}_p^\dagger \boldsymbol{B}_p \boldsymbol{s}_p} \tag{12}$$

where

$$\boldsymbol{B}_p = (\boldsymbol{S}\boldsymbol{S}^\dagger + \sigma^2 \mathbf{I})^{-1}$$

and $\boldsymbol{S} = [\boldsymbol{s}_0,\ \ldots,\ \boldsymbol{s}_{p-1},\ \boldsymbol{s}_{p+1},\ \ldots,\ \boldsymbol{s}_{K_e-1}]$ is the matrix of interfering vectors. For fixed spreading sequences, the SIR achieved at the output of the MMSE detector is defined analogous to (4) and can be written as

$$\Gamma_p = \boldsymbol{s}_p^\dagger \boldsymbol{B}_p \boldsymbol{s}_p = \boldsymbol{s}_p^\dagger (\boldsymbol{S}\boldsymbol{S}^\dagger + \sigma^2 \mathbf{I})^{-1} \boldsymbol{s}_p. \tag{13}$$

The problem then is the analysis of the above SIR in the asynchronous system, and its relation to chip/symbol synchronous systems. Clearly, the SIR is independent of the phases of the users, and henceforth, we set the phases to zero without loss of generality.

IV. MMSE Detector: SIR Analysis

Following the work in [5], [6], we model the sequences $\boldsymbol{c}_j$ to be i.i.d. random vectors and consider the large system asymptote where the number of users (K) and the processing gain (N) are scaled to infinity with $K/N \to \beta$. Now, the asymptotic analysis in [5], [6] relies on the condition that the sequence vector $\boldsymbol{s}_j$ has i.i.d. entries. For the symbol-synchronous case, we only need to consider one symbol per user, and we have $\boldsymbol{s}_j = \boldsymbol{c}_j$, $\forall j = 0,\ \ldots,\ K$. Thus, the required i.i.d. condition is immediately satisfied. In [7], the condition on $\boldsymbol{s}_j$ is relaxed to having independent entries conditioned on the delay τ_j. This requirement is satisfied in the chip-synchronous situation, since each entry in $\boldsymbol{s}_j$ is either equal to zero or an entry in the corresponding spreading sequence $\boldsymbol{c}_j$. However, when the users are completely asynchronous, the

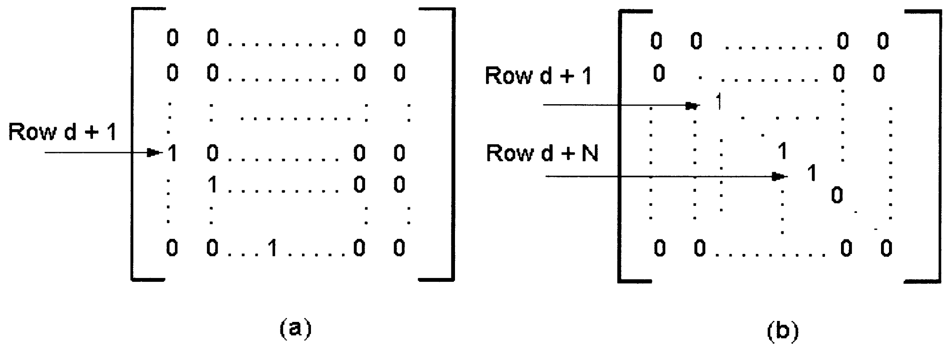

Fig. 1. Pictorial representation of (a) the matrix $\tilde{\boldsymbol{I}}(d)$; (b) the matrix $\boldsymbol{I}(d)$.

elements of $\boldsymbol{s}_j$ are neither i.i.d. nor independent when conditioned on the delay τ_j, as can be seen from (10). Hence, it appears that standard results from random matrix theory cannot be applied to compute the asymptotic SIR.

Our approach to the asynchronous problem is to consider the specific case where all the users employ the infinite duration sinc waveform $\psi^\star(t)$. As discussed in Section II, the sinc waveform is optimal for the MF detector, and use of this waveform allows us to establish an equivalence between asynchronous and (symbol) synchronous systems for the MF detector. The key property of the sinc waveform underlying this equivalence is (6). To establish a similar equivalence for the MMSE detector, the key property required of the sinc waveform is less obvious and is stated later in Lemma 1. We first give the following definitions.

Definition 1: For given M and N, and an integer d, the *chip-synchronous* matrix is defined as

$$\tilde{\boldsymbol{I}}(d) = \boldsymbol{R}(d)$$

where $\boldsymbol{R}$ is as defined in (11). Furthermore, the *partial identity matrix* of size $MN \times MN$ is defined as

$$\boldsymbol{I}(d) = \tilde{\boldsymbol{I}}(d)\tilde{\boldsymbol{I}}(d)^\dagger.$$

Since d is an integer, it can be seen from (11) that the entries of $\tilde{\boldsymbol{I}}(d)^\dagger$ are zeros except along the diagonal d, and the entries along diagonal d are all equal to 1. Here, the main diagonal is indexed as 0, and the index is positive above the main diagonal and negative below it. Consequently, $\boldsymbol{I}(d)$ is a $MN \times MN$ diagonal matrix with a string of ones along a part of the main diagonal and zeros elsewhere (see Fig. 1).

$$\boldsymbol{I}(d)[i, m] = \begin{cases} 1, & i = m \text{ and } i \in \{d+1, \ldots, d+N\} \\ & \qquad \cap \{1, \ldots, MN\} \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

Now, let $\tau_j = d_j + \gamma_j$, where τ_j is the delay of symbol j, and d_j is its integer part. By Definition 1, the vector $\tilde{\boldsymbol{I}}(d_j)\boldsymbol{c}_j$ is a length MN vector obtained when the symbol is chip-synchronous with delay d_j. Thus, if we can replace the matrix $\boldsymbol{R}(\tau_j)$ in (10) by $\tilde{\boldsymbol{I}}(d_j)$ for each j, we obtain the chip-synchronous system. To obtain an equivalence, we thus need $\boldsymbol{R}(\tau_j)$ to be close to $\tilde{\boldsymbol{I}}(d_j)$. The precise requirement is given in terms of the partial identity matrix by the following lemma.

Lemma 1: Let τ be an arbitrary real number, and let $\tau = d + \gamma$, where $d = \lfloor \tau \rfloor$ and $\gamma \in [0, 1)$. Then, the matrices $\boldsymbol{R}(\tau)\boldsymbol{R}(\tau)^\dagger$ and $\boldsymbol{I}(d)$ become equivalent as N increases, i.e.,

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \left\| \boldsymbol{R}(\tau)\boldsymbol{R}(\tau)^\dagger - \boldsymbol{I}(d) \right\| = 0$$

where $\|\boldsymbol{A}\| = [\mathrm{Tr}(\boldsymbol{A}^\dagger \boldsymbol{A})]^{\frac{1}{2}}$ is the Frobenius norm of a matrix $\boldsymbol{A}$.

Proof: The proof is somewhat cumbersome, but essentially involves getting bounds on each of the elements in $\boldsymbol{R}(\tau)\boldsymbol{R}(\tau)^\dagger - \boldsymbol{I}(d)$. In addition to (6), we make use of the fact that

$$\sum_{j=-\infty}^{\infty} \mathrm{sinc}(j+\tau)\,\mathrm{sinc}(j+\tau+q) = 0, \quad \forall \tau \text{ and any integer } q \neq 0.$$

See Appendix A for the complete proof. □

It is important to note that, as with (6), Lemma 1 is just a mathematical property of the sinc waveform, with no direct relation to the CDMA system under consideration. However, the notation used in the lemma is indeed motivated by the CDMA system: τ can be thought of as representing the delay (normalized to T_c) of a generic user in the system, with d and γ being the corresponding integer and fractional parts, respectively. Further, we note that τ can be an arbitrary function of N in Lemma 1. However, in the context of asynchronous CDMA, it is reasonable to assume that $\frac{\tau}{N}$, the delay normalized to the *symbol* interval, converges to a constant, i.e., τ is asymptotically linear in N. Finally, from the proof in Appendix A, note that $\boldsymbol{R}(\tau)\boldsymbol{R}(\tau)^\dagger$ does not go to the partial identity matrix elementwise, since some of the elements in the difference remain finite for all N. But the contribution of these elements to the Frobenius norm becomes negligible when divided by $\sqrt{N}$.

We are now in a position to provide our main result, which is that the equivalence result can indeed be obtained by using the property of the sinc waveform stated in Lemma 1. While it is possible to prove this result for a general value of M, we begin with the one-shot scenario $(M = 1, p = 0)$ for simplicity in exposition of the proof. We assume that the symbol energies of the actual (as opposed to effective) users $\mathcal{E}_k$ are bounded for $k = 0, \ldots, K$, and their empirical distribution converges to a fixed distribution in the large-system asymptote. Similarly, the actual delays normalized to the *symbol* interval, $\{\tau_k/N\}$ for $k = 0, \ldots K$, have an empirical distribution that converges to a fixed distribution.

Theorem 1: As $K, N \to \infty$ with $K/N \to \beta$, the SIR Γ_0 of the one-shot MMSE detector converges in mean square to the asymptotic SIR for the one-shot chip-synchronous system.

Proof: (Outline) The complete proof is provided in Appendix B. We summarize here the basic idea and the connection to Lemma 1. It is relatively straightforward to show that

$$\lim_{N\to\infty} \Gamma_0 = \lim_{N\to\infty} A_0^2 \frac{1}{N} \mathrm{Tr}\left\{ (\boldsymbol{S}\boldsymbol{S}^\dagger + \sigma^2 \mathbf{I})^{-1} \right\} \tag{15}$$

where the equality is in the mean-square sense. The proof then relies on a repeated application of the matrix inversion lemma to the expression in the right-hand side (RHS) of (15). In each step, the rank one matrix corresponding to effective interferer j

$$\boldsymbol{s}_j \boldsymbol{s}_j^\dagger = A_j^2 \boldsymbol{R}(\tau_j)\boldsymbol{c}_j\boldsymbol{c}_j^\dagger \boldsymbol{R}(\tau_j)^\dagger$$

is separated from the matrix $\boldsymbol{SS}^\dagger$, and the resulting perturbation of the SIR is shown to be close to a function of the matrix $\boldsymbol{R}(\tau_j)\boldsymbol{R}(\tau_j)^\dagger$. Lemma 1 is then invoked and the matrix $\boldsymbol{R}(\tau_j)\boldsymbol{R}(\tau_j)^\dagger$ is replaced by the matrix $\boldsymbol{I}(d_j) = \tilde{\boldsymbol{I}}(d_j)\tilde{\boldsymbol{I}}(d_j)^\dagger$ (see (32) and (33)). Finally, this replacement is shown to be equivalent to replacing $\boldsymbol{R}(\tau_j)\boldsymbol{c}_j$ by $\tilde{\boldsymbol{I}}(d_j)\boldsymbol{c}_j$.

Thus, the basic idea of the proof is to sequentially replace each of the asynchronous interferers' vectors with an equivalent chip-synchronous vector, and show that resulting difference is asymptotically negligible. Note that the proof does not rely on any averaging over the delays or sequences. The details are provided in Appendix B. □

The technique extends to the multi-shot scenario ($M > 1$), with an appropriate modification of the initial steps in the proof of Theorem 1.

Proposition 2: For the multishot detector, the SIR of symbol p, Γ_p, converges in mean square to the SIR for the multishot chip-synchronous system.

Proof: See Appendix C. □

For the sake of completeness, we note that the SIR for the chip-synchronous system converges in probability to a limit given by the following implicit equation, as shown in [7]:

$$\lim_{N\to\infty} \Gamma_p = \int_p^{p+1} w(x)\,dx$$

where

$$w(x) = \frac{\mathcal{E}_1}{\sigma^2 + \beta \mathrm{E}_{\mathcal{E}} \mathrm{E}_\eta\, I\left(\mathcal{E},\, \mathcal{E}_1,\, \int_{C(x,\eta)} w(z)\,dz\right)} \tag{16}$$

and the region of integration $C(x, \eta)$ is given by

$$C(x,\eta) = \begin{cases} [0,\, \eta], & x \in [0,\, \eta] \\ [\eta + m - 1,\, \eta + m], & x \in [\eta + m - 1,\, \eta + m] \\ & \text{for } m = 1, \ldots, (M-1) \\ [\eta + M - 1,\, M], & x \in [\eta + M - 1,\, M]. \end{cases}$$

Here,

$$I(\mathcal{E},\, \mathcal{E}_1,\, \Gamma) = \frac{\mathcal{E}\mathcal{E}_1}{\mathcal{E}_1 + \mathcal{E}\Gamma}$$

and the expectation is over the limiting empirical distributions of $\{\mathcal{E}_k\}$ and $\{\eta_k\}$, where $\eta_k = \frac{\tau_k}{N}$. Note that, while our convergence result is in the stronger mean-square sense, the overall convergence of the SIR for the asynchronous system to the expression in (16) is in probability, since the convergence shown in [7] is in probability.

Now, as $M \to \infty$, the SIR of the chip-synchronous system is also known to converge to the SIR for the symbol-synchronous system [7]. The equivalence result in Proposition 1 then leads us to conclude the following: *the SIR of the MMSE detector in the asynchronous system converges, as M increases, to the SIR in an equivalent symbol-synchronous system.*[4] By equivalent, we mean that all parameters, except the delays of the users, are kept the same in both systems.

The theoretical results and observations above are easily verified through numerical simulations for a finite system. In Fig. 2, the value of N is set at 32 and the average of the SIR (over spreading sequences as well as delays) is shown for the one-shot detector ($M = 1$) and for $M = 3$. The SIR with symbol-synchronous users is also shown. We see an excellent match between the asynchronous and chip-synchronous cases, and note that the average SIRs approach that in the symbol-synchronous case as M increases. Since we have proved convergence in mean square, it is also of interest to study the convergence rate of the SIR to its mean. Fig. 3 shows the ratio of the standard deviation to the

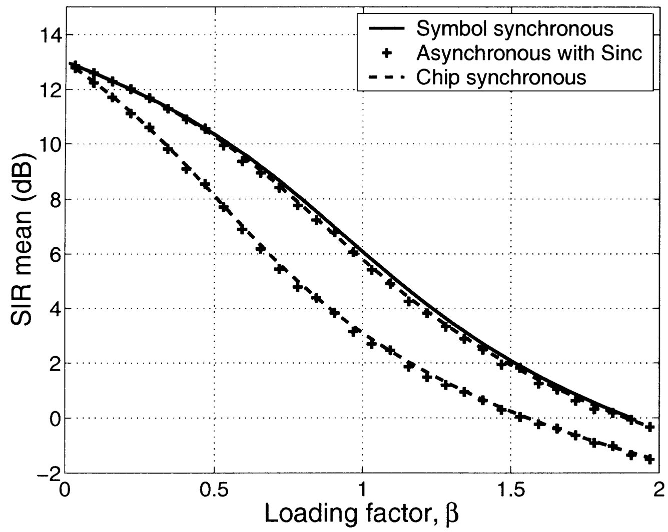

Fig. 2. Average SIR for asynchronous, chip-synchronous and symbol-synchronous systems, $N = 32$. As expected, the SIR for the asynchronous system with the sinc waveform matches that in the chip-synchronous system.

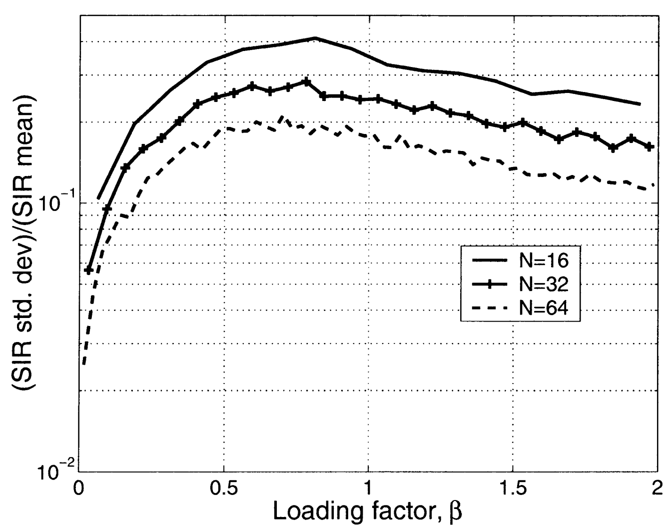

Fig. 3. Ratio of standard deviation to mean of the SIR, $N = 16, 32, 64$. As expected, the ratio decreases with increasing N, for all values of β.

mean of the SIR for different values of N. We note that, while the ratio does decrease with N for all values of β, the convergence is rather slow, with the ratio taking values of up to 0.2 when $N = 64$.

Finally, we note that the equivalence result for the MMSE detector has more general implications. Indeed, what we have proved is more fundamental than what the results for the MMSE detector indicate. Define the Stieltjes transform of $\boldsymbol{SS}^\dagger$ as [14]

$$m(z) = \frac{1}{N}\,\mathrm{Tr}\left\{(\boldsymbol{SS}^\dagger - z\boldsymbol{I})^{-1}\right\}$$

where $z \in \mathcal{C}$. Then, the equivalence is a result of the fact that $m(z)$ for the asynchronous system approaches that in the symbol-synchronous case for any z such that $\mathrm{Re}\{z\} < 0$. It follows that any performance measure which can be expressed in terms of $m(z)$, with $\mathrm{Re}\{z\} < 0$, is equal for the chip-synchronous and asynchronous cases. In particular, consider the sum of the information rates of the users when we allow for joint decoding of all the users. If we assume that the symbols have

[4]It is also interesting to note that it is when $M \to \infty$ (and not just $N \to \infty$) that the chip-MF statistics become sufficient.

an i.i.d. Gaussian distribution, the sum rate (normalized to N) is given by [15, eq. 141]

$$\begin{aligned} C_s &= \frac{1}{N}\log_2 \det\left(\boldsymbol{I} + \frac{1}{\sigma^2}\boldsymbol{S}\boldsymbol{S}^\dagger\right) \quad \text{bits per chip} \\ &= \int_0^1 \frac{1}{t}\left(1 - \frac{\sigma^2}{t}\, m\left(-\frac{\sigma^2}{t}\right)\right) dt. \end{aligned} \tag{17}$$

Thus, the sum rate can be directly related to the Stieltjes transform. To allow for an interchange of the limit and the integral when we let $N \to \infty$ in (17), we need to impose a mild sufficient condition that $\boldsymbol{c}_j^\dagger \boldsymbol{c}_j$ is bounded above for all symbols j, with a bound that is independent of N. (This condition is clearly satisfied for sequences from a finite alphabet.) Under this condition, we have the following corollary to Proposition 1.

Corollary 1: When all the users have i.i.d. Gaussian symbols, the sum rate is asymptotically equal in the chip-synchronous and asynchronous systems. Further, as $M \to \infty$, the limit (in N) of the sum rate in the asynchronous system converges to that for the synchronous system.

When the transmitters do not know the delays, asynchrony would reduce the capacity from that of the symbol-synchronous system [16], [17]. Furthermore, the i.i.d. Gaussian distribution is optimum for the symbol-synchronous system [18]. These two observations, when combined with Corollary 1, suggest that the i.i.d. Gaussian distribution could be optimum for the asynchronous system as $N \to \infty$ and then $M \to \infty$. However, we note immediately that this argument is not rigorous. We have implicitly assumed long spreading sequences, which makes the multiaccess channel time varying in addition to having memory. Further, it is not clear if the limits in M and N can be interchanged. A rigorous information-theoretic capacity analysis in the asynchronous scenario appears to be a nontrivial problem.

V. Conclusion

We have considered analysis of MMSE detection in an asynchronous system with random spreading. Under the assumption that the chip waveform is the ideal sinc function, we have shown that the SIR is the same as that in an equivalent chip-synchronous system, for any fixed window size. As the window size goes to infinity, our results imply that the SIR is the same as that in an equivalent symbol-synchronous system.

Now, the sinc chip waveform maximizes the processing gain for a given symbol rate and bandwidth. We conjecture that this fact would make the sinc waveform optimal for the MMSE detector over all chip waveforms, in the sense of maximizing the SIR under equal symbol rate and bandwidth constraints. Furthermore, practical CDMA standards use chip waveforms that have an approximately flat spectrum. Hence, a system employing the sinc waveform is a natural benchmark for asynchronous analyses. Since we have proved that such a system is equivalent to a synchronous system, our results provide a justification for synchronous random sequence analyses for asynchronous band-limited CDMA systems.

To formally establish the optimality of the sinc waveform, it may be necessary to analyze the SIR with a general chip waveform. This appears to be a more difficult problem and could be a subject for further study. It would also be of interest to study equivalence for other detectors, notably the decorrelating detector. While the decorrelating detector can be obtained as the limit of the MMSE detector as $\sigma^2 \to 0$, our proof relies on bounds involving $\frac{1}{\sigma^2}$ and is not applicable for the decorrelator. Since the equivalence result for the MMSE detector stems from the convergence of the Stieltjes transform of the covariance matrix $\boldsymbol{S}\boldsymbol{S}^\dagger$, it is possible that the equivalence holds more generally, perhaps for the class of detectors considered in [19].

Appendix A
Proof of Lemma 1

We make use of the following simple result in the proof.

Lemma 2: Let $g(x)$ be a positive, integrable, and monotone-decreasing function. Then, for integers $a,\, b > 1$, a fraction $\gamma \in [0,\, 1)$

$$\sum_{k=a}^{b} g(k+\gamma) \le \sum_{k=a}^{b} g(k) \le \int_{a-1}^{b} g(x)\, dx.$$

In particular, for $g(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$, we have, respectively,

$$\sum_{k=a}^{b} \frac{1}{k+\gamma} \le \log\left(\frac{b}{a-1}\right) = \log\left(1 + \frac{b-a+1}{a-1}\right) \tag{18}$$

$$\sum_{k=a}^{b} \frac{1}{(k+\gamma)^2} \le \frac{1}{a-1} - \frac{1}{b} \le \frac{1}{a-1} \tag{19}$$

where $a \ge 2$ and $b > a$.

Proof of Lemma 1: Let $\boldsymbol{X} = \boldsymbol{R}(\tau)\boldsymbol{R}(\tau)^\dagger$, $d = \lfloor \tau \rfloor$, and $\gamma = \tau - d \in [0,\, 1)$. Throughout the proof, N is kept fixed, and hence, the dependence of d, γ and $\boldsymbol{X}$ on N is suppressed. We have

$$\begin{aligned} \boldsymbol{X}[i,\, m] &= \sum_{k=1}^{N} \boldsymbol{R}(\tau)[i,\, k]\boldsymbol{R}(\tau)[m,\, k] \\ &= \sum_{k=1}^{N} \text{sinc}(d+\gamma+k-i)\text{sinc}(d+\gamma+k-m) \end{aligned}$$

and we need to show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i,\, m=1}^{MN} (\boldsymbol{X}[i,\, m] - \boldsymbol{I}(d)[i,\, m])^2 = 0.$$

The Frobenius norm is estimated along the diagonals of $\boldsymbol{X} - \boldsymbol{I}(d)$. The elements of $\boldsymbol{X}$ along diagonal q are given by

$$\boldsymbol{X}[i,\, i+q] = \sum_{k=a}^{b} \text{sinc}(k+\gamma)\text{sinc}(k+\gamma-q) := f(a,\, b,\, q)$$

where $a = d - i + 1$ and $b = d - i + N$. We can write out $f(a,\, b,\, q)$ as

$$\begin{aligned} f(a,\, b,\, q) &= \frac{1}{\pi^2} \sum_{k=a}^{b} \frac{(-1)^k(-1)^{k-q}\sin^2(\pi\gamma)}{(k+\gamma)(k+\gamma-q)} \\ &= c_\gamma (-1)^q \sum_{k=a}^{b} \frac{1}{(k+\gamma)(k+\gamma-q)} \end{aligned}$$

where $c_\gamma = \frac{\sin^2(\pi\gamma)}{\pi^2}$.

Since $\boldsymbol{X} - \boldsymbol{I}(d)$ is a symmetric matrix, we only need to consider the upper half corresponding to $q = 0, \ldots, MN-1$ and $i = 1, \ldots, MN - q$. Let $h(q)$ be the contribution of diagonal q to $\|\boldsymbol{X} - \boldsymbol{I}(d)\|^2$, i.e.,

$$\begin{aligned} h(q) &= \sum_{i=1}^{MN-q} (\boldsymbol{X}[i,\, i+q] - \boldsymbol{I}(d)[i,\, i+q])^2 \\ &= \sum_{i=1}^{MN-q} |f(a,\, b,\, q) - \boldsymbol{I}(d)[i,\, i+q]|^2 \end{aligned} \tag{20}$$

and we would like to show

$$\frac{1}{N} \sum_{q=0}^{MN-1} h(q) \to 0.$$

We study $h(q)$ for three different cases: i) the main diagonal, $q = 0$; ii) the first N off-diagonals, $q = 1, \ldots, N$; iii) the remaining diagonals, $q > N$.

A. Main Diagonal Elements, $q = 0$

Along the main diagonal, the entries are positive and close to 1 between $i = d + 1$ and $i = d + N$, and close to 0 otherwise. Hence, we further split the diagonal elements into three groups: $i \le d$, $i \in \{d+1, \ldots, d+N\}$, and $i > d+N$. Note that some of the sets may be empty depending on the value of d, since we also require i to be between 1 and MN. However, this does not affect the analysis, and we retain all three groups.
For $i < d$, we have $a = d - i + 1 > 1$ and, using (19)

$$|f(a, b, 0)| = c_\gamma \sum_{k=a}^{b} \frac{1}{(k+\gamma)^2} \le \frac{c_\gamma}{a-1}. \tag{21}$$

For $i > d + N + 2$, we have $b = d - i + N < -2$ and

$$|f(a, b, 0)| = c_\gamma \sum_{k=|b|}^{|a|} \frac{1}{(k-\gamma)^2} \le c_\gamma \frac{1}{|b+2|}. \tag{22}$$

For the intermediate index set $i \in \{d+3, \ldots, d+N-1\}$, we have $a < 0$ and $b > 0$ so that

$$|1 - f(a, b, 0)| = c_\gamma \left| \sum_{k=-\infty}^{a-1} \frac{1}{(k+\gamma)^2} + \sum_{k=b+1}^{\infty} \frac{1}{(k+\gamma)^2} \right| \tag{23}$$

where we have used the fact that

$$\sum_{k=-\infty}^{\infty} \operatorname{sinc}^2(k+\gamma) = c_\gamma \sum_{k=-\infty}^{\infty} \frac{1}{(k+\gamma)^2} = 1, \qquad \forall \gamma.$$

Consequently

$$|1 - f(a, b, 0)| \le c_\gamma \left(\frac{1}{|a+1|} + \frac{1}{b} \right).$$

Note that we have ignored a few terms around the transition points $i = d + 1$ and $i = d + N$ in the above estimation, specifically

$$i \in \{d, d+1, d+2\}$$

and

$$i \in \{d+N, d+N+1, d+N+2\}.$$

The number of such terms remains finite as N increases and hence, their contribution can be bounded by an $O(1)$ term.

Now, recall from Definition 1 and (14) that $\boldsymbol{I}(d)$ is a diagonal matrix with a string of ones from $\boldsymbol{I}(d)[d+1, d+1]$ to $\boldsymbol{I}(d)[d+N, d+N]$. Hence, combining (21)–(23), and using (20), we have

$$\begin{aligned} h(0) &= \sum_{i=1}^{d-1} |f(a, b, 0)|^2 + \sum_{i=d+3}^{d+N-1} |1 - f(a, b, 0)|^2 \\ &\quad + \sum_{i=d+N+3}^{MN} |f(a, b, 0)|^2 + O(1) \\ &= c_\gamma^2 \sum_{i=1}^{d-1} \frac{1}{(d-i)^2} + c_\gamma^2 \sum_{i=d+3}^{d+N-1} \left[\frac{1}{(i-d)} + \frac{1}{(d-i+N)} \right]^2 \\ &\quad + c_\gamma^2 \sum_{i=d+N+3}^{MN} \frac{1}{(i-N-d-2)^2} + O(1) \\ &\le c_\gamma^2 \left\{ \sum_{j=1}^{d-1} \frac{1}{j^2} + 2 \sum_{j=3}^{N-1} \left[\frac{1}{j^2} + \frac{1}{(N-j)^2} \right] \right. \\ &\qquad \left. + \sum_{j=1}^{MN-N-d-2} \frac{1}{j^2} + O(1) \right\} \sim O(1). \end{aligned}$$

Thus, the main diagonal elements yield an $O(1)$ term to $\operatorname{Tr}\{(\boldsymbol{X} - \boldsymbol{I}(d))^2\}$, and the contribution to $\frac{1}{\sqrt{N}} \|\boldsymbol{X} - \boldsymbol{I}(d)\|$ goes to zero with N.

B. Off-Diagonal Elements, $q > 0$ and $q \le N$

Broadly, the off-diagonal elements are finite but small. For $q > 0$, we can write $f(a, b, q)$ as

$$\begin{aligned} f(a, b, q) &= c_\gamma (-1)^q \sum_{k=a}^{b} \frac{1}{(k+\gamma)(k+\gamma-q)} \\ &= c_\gamma \frac{(-1)^q}{q} \sum_{k=a}^{b} \left[\frac{1}{k+\gamma-q} - \frac{1}{k+\gamma} \right]. \end{aligned}$$

Since $b - a = N - 1$, there are always N terms in the summation. Since $q \le N$, some of the terms cancel to yield

$$f(a, b, q) = c_\gamma \frac{(-1)^q}{q} \left(\sum_{k=a-q}^{a-1} \frac{1}{k+\gamma} - \sum_{k=b-q+1}^{b} \frac{1}{k+\gamma} \right).$$

Note that i goes from 1 to $MN - q$ as we move down diagonal q. Hence, $a\ (=d - i + 1)$ goes from d down to $d + q + 1 - MN$. We again consider different groups of elements along diagonal q, even though some of the groups may be empty for given values of d and q.

Case 1, $a - q > 1$: We have $b - q + 1 > 0$, and

$$\begin{aligned} |f(a, b, q)| &\le \frac{c_\gamma}{q} \sum_{k=a-q}^{a-1} \frac{1}{k+\gamma} \\ &\le \frac{c_\gamma}{q} \log\left(1 + \frac{q}{a-q-1}\right) \end{aligned}$$

where we have used (18). The contribution of this group of elements to $h(q)$ is

$$\begin{aligned} \sum_{a:\, a-q>1} |f(a, b, q)|^2 &= \sum_{j=2}^{d-q} |f(j+q, j+q+N-1, q)|^2 \\ &\le \frac{c_\gamma^2}{q^2} \sum_{j=2}^{d-q} \log^2\left(1 + \frac{q}{j}\right). \end{aligned}$$

Now, for any $J \ge 1$

$$\begin{aligned} &\frac{1}{q^2} \sum_{j=1}^{J} \log^2\left(1 + \frac{q}{j}\right) \\ &\quad \le \frac{1}{q^2} \sum_{j=1}^{q} \log^2\left(1 + \frac{q}{j}\right) + \frac{1}{q^2} \sum_{j=q+1}^{\infty} \log^2\left(1 + \frac{q}{j}\right) \\ &\quad \le \frac{1}{q} \log^2(1+q) + \frac{1}{q^2} \sum_{j=q+1}^{\infty} \frac{q^2}{j^2} \\ &\quad \le \frac{1}{q} \log^2(1+q) + \frac{1}{q} \end{aligned}$$

where we have again used (19) along with the monotonicity of the $\log$ function and the inequality $\log(1 + x) \le x$. Noting that $q \le N$, we can get a loose estimate as

$$\sum_{a:\, a-q>1} |f(a, b, q)|^2 \le 2c_\gamma^2 \frac{\log^2(N+1)}{q}.$$

Finally, summing up across the diagonals ($q = 1$ to N), we see that the total contribution of such elements can be at most $O(\log^3 N)$, which goes to zero when we divide by N.

Case 2, $a - q < 0$ but $a - 1 > 0$: We have $b - q + 1,\ b > 0$. Hence,

$$|f(a, b, q)| \le \frac{c_\gamma}{q} \left| \sum_{k=a-q}^{a-1} \frac{1}{k+\gamma} \right| + \frac{c_\gamma}{q} \left| \sum_{k=b-q+1}^{b} \frac{1}{k+\gamma} \right|.$$

The second term can be bounded by a logarithmic quantity as in Case 1, since the summation is over positive indexes. The summation in the first term is over both positive and negative indexes. Hence, we distinguish between those indexes that have a negative counterpart and those that do not. With

$$a_0 = \min(|a-q|,\, |a-1|)$$
$$a_1 = \max(|a-q|,\, |a-1|)$$

we have

$$\begin{aligned}\frac{c_\gamma}{q}\left|\sum_{k=a-q}^{a-1}\frac{1}{k+\gamma}\right| &\le \frac{c_\gamma}{q}\left|\sum_{k=-a_0}^{a_0}\frac{1}{k+\gamma}\right| + \frac{c_\gamma}{q}\left|\sum_{k=a_0+1}^{a_1}\frac{1}{k\pm\gamma}\right| \\ &= \frac{c_\gamma}{q\gamma} + \frac{c_\gamma}{q}\sum_{k=1}^{a_0}\frac{2\gamma}{k^2-\gamma^2} + \frac{c_\gamma}{q}\left|\sum_{k=a_0+1}^{a_1}\frac{1}{k\pm\gamma}\right| \\ &\le \frac{f_\gamma}{q} + \frac{c_\gamma}{q}\log\left(1+\frac{q}{j}\right) \end{aligned} \tag{24}$$

for some $j > 0$ and a constant f_γ independent of q or N. (If $|a-q| < |a-1|$, the sign for γ is $+$, and $-$ otherwise.) The sum (over j) of the squares of the second term in (24) can be bounded as in Case 1. Since we have at most q terms along diagonal q that fall under Case 2 considered here, the sum of the squares of the first terms in (24) is less than

$$q\,\frac{f_\gamma^2}{q^2} = \frac{f_\gamma^2}{q}.$$

Hence, summing across the diagonals ($q = 1$ to N) gives an estimate of the $O(\log N)$ from this set of terms, which again goes to zero when divided by N.

We have now covered all the bounding techniques involved for $q \le N$. The arguments can be carried through to other groups of elements along each diagonal, e.g., $a \le 1$ but $b - q + 1 > 0$ etc., and we get an overall estimate of $O(\frac{\log^3 N}{N})$ from the first N diagonals.

C. Off-Diagonal Elements, $q > N$

When $q > N$, we have

$$f(a,\, b,\, q) = c_\gamma\,\frac{(-1)^q}{q}\sum_{k=a}^{b}\left[\frac{1}{k+\gamma-q} - \frac{1}{k+\gamma}\right]$$

and the terms do not cancel. Instead, each element on diagonal q can be bounded as

$$|f(a,\, b,\, q)| \le \frac{c_\gamma}{q}\left|\sum_{k=a-q}^{b-q}\frac{1}{k+\gamma}\right| + \frac{c_\gamma}{q}\left|\sum_{k=a}^{b}\frac{1}{k+\gamma}\right|.$$

The number of terms in each summation is N. Hence the estimation techniques for the case $q \le N$ in Appendix A, part B above would hold, with the modification that $\log(1+\frac{q}{j})$ would be replaced by $\log(1+\frac{N}{j})$. Consequently, the contribution to the Frobenius norm would involve, for some $J \ge 1$,

$$\begin{aligned}\frac{1}{q^2}\sum_{j=1}^{J}\log^2\left(1+\frac{N}{j}\right) &\le \frac{N}{q^2}\log^2(1+N) + \frac{N}{q^2} \\ &\le \frac{\log^2(1+N)}{q} + \frac{1}{q}\end{aligned}$$

where the last step follows since $q > N$. Finally, summation of $1/q$ across the diagonals from $N+1$ to MN is bounded by $\log(MN/N) = \log M$, which would yield an estimate of $O(\log^2 N)$ on $\sum_{q>N} h(q)$.

Hence, the contribution of this part also goes to zero when divided by N, and we have the desired result in Lemma 1

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\,\|\boldsymbol{X} - \boldsymbol{I}(d)\| = 0.$$

Appendix B
Proof of Theorem 1

We will need the following result from [20].

Lemma 3 [20, Lemma 14]: Suppose $\nu_1,\,\ldots,\,\nu_{MN}$ are i.i.d. random variables, each with zero mean, variance $1/N$, and a finite fourth moment. Let $\boldsymbol{B}$ be an $MN \times MN$ constant Hermitian matrix. Define the vector

$$\boldsymbol{s} = [a_1\nu_1, \ldots, a_1\nu_N, a_2\nu_{N+1}, \ldots, a_M\nu_{N(M-1)+1}, \ldots, a_M\nu_{MN}]^\top$$

where $a_1,\,\ldots,\,a_M$ are deterministic and real-valued. Then

$$E\boldsymbol{s}^\dagger\boldsymbol{B}\boldsymbol{s} = \frac{1}{N}\sum_{m=1}^{M}|a_m|^2 T_m(\boldsymbol{B})$$

and

$$\mathrm{Var}\left(\boldsymbol{s}^\dagger\boldsymbol{B}\boldsymbol{s}\right) \le C_1\frac{\rho(\boldsymbol{B})^2}{N}$$

where $\rho(\boldsymbol{B})$ is the spectral radius (or maximum eigenvalue) of $\boldsymbol{B}$, the constant C_1 depends only on $\{a_m\}$ and the fourth moment of ν_1, and

$$T_m(\boldsymbol{B}) = \sum_{i=(m-1)N+1}^{mN} B_{ii}.$$

Proof of Theorem 1: To make it more readable, we number the key steps in the following proof.

1) Reduction to the Trace: We begin with the SIR expression

$$\Gamma_0 = \boldsymbol{s}_0^\dagger\left(\boldsymbol{S}\boldsymbol{S}^\dagger + \sigma^2\mathbf{I}\right)^{-1}\boldsymbol{s}_0$$

where $\boldsymbol{s}_0 = A_0\boldsymbol{R}(\tau_0)\boldsymbol{c}_0$ and $\boldsymbol{S}$ is the $N \times 2K$ matrix corresponding to the effective spreading sequence of the interfering vectors. Since we have assumed $\tau_0 = 0$, $\boldsymbol{R}(\tau_0)$ is the identity matrix. Hence,

$$\Gamma_0 = A_0^2\boldsymbol{c}_0^\dagger\boldsymbol{B}\boldsymbol{c}_0$$

where $\boldsymbol{B} = (\boldsymbol{S}\boldsymbol{S}^\dagger + \sigma^2\mathbf{I})^{-1}$. The vector $\boldsymbol{c}_0$ is a vector of length N with i.i.d. entries of zero mean, variance $1/N$, and a finite fourth moment. Also, note that $|\boldsymbol{B}| \le \frac{1}{\sigma^2}$. Hence, applying Lemma 3 with $M = 1$ and $a_1 = 1$, $\boldsymbol{c}_0^\dagger\boldsymbol{B}\boldsymbol{c}_0$ converges in mean square to $\frac{1}{N}T_1\{\boldsymbol{B}\} = \frac{1}{N}\mathrm{Tr}\{\boldsymbol{B}\}$. We denote this convergence as

$$\boldsymbol{c}_0^\dagger\boldsymbol{B}\boldsymbol{c}_0 \overset{m.s.}{\to} \frac{1}{N}\mathrm{Tr}\{\boldsymbol{B}\}.$$

Thus, we need to prove that $\frac{1}{N}\mathrm{Tr}\{\boldsymbol{B}\}$ converges to the same limit as in the chip-synchronous case. In other words, we need to prove that the fractional delays of the interfering symbols can be set to zero without affecting the limit. We prove this by applying the matrix inversion lemma for each of the $2K$ interfering vectors (indexed from $j = 1$ to $2K$) in an iterative manner.

2) Application of the Matrix Inversion Lemma to a Single Interferer: Let

$$\boldsymbol{B}_{(1)} = (\boldsymbol{S}_{(1)}\boldsymbol{S}_{(1)}^\dagger + \sigma^2\mathbf{I})^{-1}$$

where the matrix $\boldsymbol{S}_{(1)}$ is formed by removing the first interfering symbol's vector $\boldsymbol{s}_1$ from $\boldsymbol{S}$. Also, let $\boldsymbol{R}_j = \boldsymbol{R}(\tau_j)$ for $j = 1,\,\ldots,\,2K$.

Then, by the matrix inversion lemma

$$\begin{aligned}\mathrm{Tr}\{\boldsymbol{B}\} &= \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}\right\} - \frac{\boldsymbol{s}_1^\dagger \boldsymbol{B}_{(1)}^2 \boldsymbol{s}_1}{1+\boldsymbol{s}_1^\dagger \boldsymbol{B}_{(1)} \boldsymbol{s}_1} \\ &= \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}\right\} - \frac{A_1^2 \boldsymbol{c}_1^\dagger \boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)}^2 \boldsymbol{R}_1 \boldsymbol{c}_1}{1+A_1^2 \boldsymbol{c}_1^\dagger \boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)} \boldsymbol{R}_1 \boldsymbol{c}_1} \\ &= \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}\right\} - \frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)}^2 \boldsymbol{R}_1\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)} \boldsymbol{R}_1\right\}} + \Delta f_1\end{aligned}$$

where

$$\Delta f_1 = \frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)}^2 \boldsymbol{R}_1\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)} \boldsymbol{R}_1\right\}} - \frac{A_1^2 \boldsymbol{c}_1^\dagger \boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)}^2 \boldsymbol{R}_1 \boldsymbol{c}_1}{1+A_1^2 \boldsymbol{c}_1^\dagger \boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)} \boldsymbol{R}_1 \boldsymbol{c}_1}. \tag{25}$$

Furthermore, if $\boldsymbol{X}_1 = \boldsymbol{R}_1\boldsymbol{R}_1^\dagger$, we note that the first term in the RHS of (25) can be written as

$$\begin{aligned}&\frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)}^2 \boldsymbol{R}_1\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{R}_1^\dagger \boldsymbol{B}_{(1)} \boldsymbol{R}_1\right\}} \\ &\quad= \frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}^2 \boldsymbol{X}_1\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)} \boldsymbol{X}_1\right\}} \\ &\quad= \frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}^2 \boldsymbol{I}(d_1)\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)} \boldsymbol{I}(d_1)\right\}} - \Delta g_1\end{aligned}$$

where

$$\Delta g_1 = \frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}^2 \boldsymbol{I}(d_1)\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)} \boldsymbol{I}(d_1)\right\}} - \frac{1}{N}\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}^2 \boldsymbol{X}_1\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)} \boldsymbol{X}_1\right\}}. \tag{26}$$

Finally, we define

$$\begin{aligned}\Delta \tilde{f}_1 = \frac{1}{N}&\frac{A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)}^2 \boldsymbol{I}(d_1)\right\}}{1+\frac{1}{N}A_1^2\, \mathrm{Tr}\left\{\boldsymbol{B}_{(1)} \boldsymbol{I}(d_1)\right\}} \\ &- \frac{A_1^2 \tilde{\boldsymbol{c}}_1^\dagger \tilde{\boldsymbol{I}}(d_1)^\dagger \boldsymbol{B}_{(1)}^2 \tilde{\boldsymbol{I}}(d_1)\tilde{\boldsymbol{c}}_1}{1+A_1^2 \tilde{\boldsymbol{c}}_1^\dagger \tilde{\boldsymbol{I}}(d_1)^\dagger \boldsymbol{B}_{(1)} \tilde{\boldsymbol{I}}(d_1)\tilde{\boldsymbol{c}}_1}\end{aligned} \tag{27}$$

where $\tilde{\boldsymbol{c}}_1$ is a vector of length N independent of, and identically distributed as $\boldsymbol{c}_1$. Note that the definition of $\Delta\tilde{f}_1$ is similar to that of Δf in (25), except that $\boldsymbol{c}_1$ is replaced by $\tilde{\boldsymbol{c}}_1$ and $\boldsymbol{R}_1$ is replaced by $\tilde{\boldsymbol{I}}(d_1)$. Putting the above equations together, we have

$$\begin{aligned}\frac{1}{N}\mathrm{Tr}\{\boldsymbol{B}\} &= \frac{1}{N}\mathrm{Tr}\left\{\boldsymbol{B}_{(1)}\right\} - \frac{1}{N}\frac{A_1^2 \tilde{\boldsymbol{c}}_1^\dagger \tilde{\boldsymbol{I}}(d_1)^\dagger \boldsymbol{B}_{(1)}^2 \tilde{\boldsymbol{I}}(d_1)\tilde{\boldsymbol{c}}_1}{1+A_1^2 \tilde{\boldsymbol{c}}_1^\dagger \tilde{\boldsymbol{I}}(d_1)^\dagger \boldsymbol{B}_{(1)} \tilde{\boldsymbol{I}}(d_1)\tilde{\boldsymbol{c}}_1} \\ &\quad + \frac{1}{N}\left[\Delta f_1 + \Delta g_1 - \Delta\tilde{f}_1\right] \\ &= \frac{1}{N}\mathrm{Tr}\left\{\tilde{\boldsymbol{B}}_{(1)}\right\} + \frac{1}{N}\left[\Delta f_1 + \Delta g_1 - \Delta\tilde{f}_1\right]\end{aligned}$$

where $\tilde{\boldsymbol{B}}_{(1)} = (\tilde{\boldsymbol{S}}_{(1)}\tilde{\boldsymbol{S}}_{(1)}^\dagger + \sigma^2\mathbf{I})^{-1}$ and $\tilde{\boldsymbol{S}}_{(1)}$ is obtained by replacing the first column in $\boldsymbol{S}$ by a corresponding chip synchronous vector $A_1\tilde{\boldsymbol{I}}(d_1)\tilde{\boldsymbol{c}}_1$.

3) Extension to All Interferers: Repeating the above procedure for each of the $2K$ vectors in $\boldsymbol{S}$, we have

$$\begin{aligned}\frac{1}{N}\mathrm{Tr}\{\boldsymbol{B}\} &= \frac{1}{N}\mathrm{Tr}\{\tilde{\boldsymbol{B}}\} + \frac{1}{N}\sum_{j=1}^{2K}\left[\Delta f_j + \Delta g_j - \Delta\tilde{f}_j\right] \\ &= \frac{1}{N}\mathrm{Tr}\{\tilde{\boldsymbol{B}}\} + \epsilon_N\end{aligned}$$

where Δf_j, Δg_j, and $\Delta\tilde{f}_j$ are defined at step j, analogous to the respective definitions in (25)–(27). Note that $\boldsymbol{B}_{(j)}$ is obtained by removing column j from $\tilde{\boldsymbol{B}}_{(j-1)}$, with $\tilde{\boldsymbol{B}}_{(0)} = \boldsymbol{B}$ and $\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}_{(2K)}$. Thus, $\tilde{\boldsymbol{B}}$ is the matrix formed when all the $2K$ interfering symbols are chip-synchronous.

Now, $A_0^2\,\frac{1}{N}\,\mathrm{Tr}\{\boldsymbol{B}\}$ and $A_0^2\,\frac{1}{N}\,\mathrm{Tr}\{\tilde{\boldsymbol{B}}\}$ converge to the SIRs in the asynchronous and chip-synchronous cases, respectively. Hence, the remaining task is to show that ϵ_N converges to zero in mean square as $N \to \infty$.

4) Bounding $|\Delta f_j|$ and $|\Delta\tilde{f}_j|$: From (25), we have

$$\Delta f_j = \frac{1}{N}\frac{\mathrm{Tr}\{\boldsymbol{Y}_j\}}{1+\frac{1}{N}\mathrm{Tr}\{\boldsymbol{Z}_j\}} - \frac{\boldsymbol{c}_j^\dagger\boldsymbol{Y}_j\boldsymbol{c}_j}{1+\boldsymbol{c}_j^\dagger\boldsymbol{Z}_j\boldsymbol{c}_j}$$

where $\boldsymbol{Y}_j = A_j^2\boldsymbol{R}_j^\dagger\boldsymbol{B}_{(j)}^2\boldsymbol{R}_j$ and $\boldsymbol{Z}_j = A_j^2\boldsymbol{R}_j^\dagger\boldsymbol{B}_{(j)}\boldsymbol{R}_j$. Consequently, by Lemma 3

$$\boldsymbol{c}_j^\dagger\boldsymbol{Y}_j\boldsymbol{c}_j \overset{m.s.}{\to} \frac{1}{N}\mathrm{Tr}\{\boldsymbol{Y}_j\} \quad \text{and} \quad \boldsymbol{c}_j^\dagger\boldsymbol{Z}_j\boldsymbol{c}_j \overset{m.s.}{\to} \frac{1}{N}\mathrm{Tr}\{\boldsymbol{Z}_j\}. \tag{28}$$

Note that the application of Lemma 3 requires that the spectral radius of $\boldsymbol{Y}_j$ and $\boldsymbol{Z}_j$ be uniformly bounded. Assuming the symbol energies are uniformly bounded, this can be verified as follows:

$$\begin{aligned}\rho(\boldsymbol{Z}_j) &= A_j^2\rho\left(\boldsymbol{R}_1^\dagger\boldsymbol{B}_{(1)}\boldsymbol{R}_1\right) = A_j^2\rho\left(\boldsymbol{B}_{(1)}\boldsymbol{R}_1\boldsymbol{R}_1^\dagger\right) \\ &\le A_j^2\rho\left(\boldsymbol{B}_{(1)}\right)\,\rho(\boldsymbol{R}_1\boldsymbol{R}_1^\dagger) \\ &\le \frac{A_j^2}{\sigma^2}\,\rho(\boldsymbol{R}_1^\dagger\boldsymbol{R}_1) \le \frac{A_j^2}{\sigma^2}.\end{aligned}$$

The last step follows since

$$\rho(\boldsymbol{R}_1^\dagger\boldsymbol{R}_1) = \max_{\boldsymbol{c}_1}\frac{\boldsymbol{c}_1^\dagger\boldsymbol{R}_1^\dagger\boldsymbol{R}_1\boldsymbol{c}_1}{\boldsymbol{c}_1^\dagger\boldsymbol{c}_1} \tag{29}$$

and the numerator can be interpreted as the energy of the projection of the underlying continuous time signal onto the chip-MF basis functions (see (10)). The same argument holds for $\boldsymbol{Y}_j$ as well.

Now, the convergence in (28), along with the facts that the spectral norm of $\boldsymbol{Y}_j$ is bounded and the function $\frac{1}{1+y} \le 1$ for $y \ge 0$, imply the mean-square convergence of Δf_j to zero. Specifically, we have

$$E|\Delta f_j|^2 \le \frac{C_1A_j^4}{N} \tag{30}$$

where C_1 is a constant independent of j as well. The same bound would also hold for $\mathrm{E}|\Delta\tilde{f}_j|^2$.

5) Bounding $|\Delta g_j|$: The final estimate we require is for Δg_j. From (26), we have

$$\Delta g_j = \frac{1}{N}\frac{A_j^2\,\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}^2\boldsymbol{I}(d_j)\right\}}{1+\frac{1}{N}A_j^2\,\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}\boldsymbol{I}(d_j)\right\}} - \frac{1}{N}\frac{A_j^2\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}^2\boldsymbol{X}_j\right\}}{1+A_j^2\,\frac{1}{N}\,\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}\boldsymbol{X}_j\right\}}$$

where d_j is the integer part of τ_j. In general, since $\mathrm{Tr}\{\boldsymbol{AX}\}$ is an inner product for $N\times N$ matrices $\boldsymbol{A}$ and $\boldsymbol{X}$, we have

$$|\mathrm{Tr}\{\boldsymbol{A}(\boldsymbol{X}-\boldsymbol{Y})\}| \le \|\boldsymbol{A}\|\,\|\boldsymbol{X}-\boldsymbol{Y}\| \le \sqrt{N}\,\rho(\boldsymbol{A})\|\boldsymbol{X}-\boldsymbol{Y}\|$$

where $\|.\|$ denotes the Frobenius norm. Consequently

$$\begin{aligned}&\left|\frac{1}{N}\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}^2\boldsymbol{I}(d_j)\right\} - \frac{1}{N}\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}^2\boldsymbol{X}_j\right\}\right| \\ &\qquad\le \frac{1}{\sigma^4}\,\frac{1}{\sqrt{N}}\,\|\boldsymbol{X}_j - \boldsymbol{I}(d_j)\| \\ &\left|\frac{1}{N}\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}\boldsymbol{I}(d_j)\right\} - \frac{1}{N}\mathrm{Tr}\left\{\boldsymbol{B}_{(j)}\boldsymbol{X}_j\right\}\right| \\ &\qquad\le \frac{1}{\sigma^2}\,\frac{1}{\sqrt{N}}\,\|\boldsymbol{X}_j - \boldsymbol{I}(d_j)\|.\end{aligned} \tag{31}$$

Again, since the function $\frac{1}{1+y}$, the random variable $\frac{1}{N}\mathrm{Tr}\{\boldsymbol{B}_{(j)}^2\boldsymbol{I}(d_j)\}$, and the symbol energies A_j^2 are all bounded, we have

$$|\Delta g_j| \le C_2A_j^2\,\frac{1}{\sqrt{N}}\,\|\boldsymbol{X}_j - \boldsymbol{I}(d_j)\| \tag{32}$$

for some constant C_2 independent of j as well. We now apply Lemma 1 to note that the RHS goes to zero for all realizations of the delays $\tau_j = d_j + \gamma_j$. Hence, we have that Δg_j converges to zero in mean square for all j. More precisely, from the proof of Lemma 1

$$\mathrm{E}\,|\Delta g_j|^2 \le \tilde{C}_2 A_j^4 \frac{\log^3 N}{N} \tag{33}$$

for some constant $\tilde{C}_2$. Finally, using (30) and (33), we have

$$\begin{aligned}\mathrm{E}\,|\epsilon_N|^2 &\le \frac{3}{N}\sum_{j=1}^{2K}\left(\mathrm{E}\,|\Delta f_j|^2 + \mathrm{E}\,|\Delta g_j|^2 + \mathrm{E}\,|\Delta \tilde{f}_j|^2\right) \\ &\le 3\left(\frac{2C_1}{N} + \tilde{C}_2\frac{\log^3 N}{N}\right)\frac{1}{N}\sum_{j=1}^{2K} A_j^4 \\ &\to 0, \text{ as } N \to \infty.\end{aligned}$$

The proof of Theorem 1 is complete.

Appendix C
Proof of Proposition 1

The proof of Theorem 1 relied on the one-shot assumption most importantly at the first step, *viz.*, in reducing the SIR Γ_0 to the *trace* of the matrix $\boldsymbol{B}$. When $M > 1$, the SIR for symbol p is given by

$$\Gamma_p = \boldsymbol{s}_p^\dagger(\boldsymbol{S}\boldsymbol{S}^\dagger + \sigma^2\mathbf{I})^{-1}\boldsymbol{s}_p.$$

where the matrix $\boldsymbol{S}$ excludes the vector $\boldsymbol{s}_p$. Now, the symbols of the desired user correspond to $m = 0, \ldots, M-1$. The corresponding spreading vectors $\boldsymbol{s}_m$ are of length MN and can be written down as $[0, \ldots, 0, \boldsymbol{c}_m^\top, 0, \ldots, 0]$, where the nonzero entries go from the indexes $mN+1$ to $(m+1)N$ for symbol m, $m = 0, \ldots, M-1$. Hence, applying Lemma 3 with $d_p = 1$ and $d_m = 0$ for all $m \neq p$, the SIR for symbol p is given by

$$\Gamma_p \overset{m.s.}{\to} \frac{A_p^2}{N} T_p(\boldsymbol{B}).$$

Thus, the SIR reduces to a partial diagonal sum and not trace of the matrix, and we would like to show that this partial sum is asymptotically equal to that in the chip-synchronous case. Note that, since we are already synchronized to user 1, the spreading vectors in $\boldsymbol{S}$ that correspond to the desired user need not be modified.

For any M, the first interfering symbol corresponds to the index M, and has the effective spreading vector $\boldsymbol{s}_M$. Define

$$\boldsymbol{B}_{(M)} = (\boldsymbol{S}_{(M)}\boldsymbol{S}_{(M)}^\dagger + \sigma^2\boldsymbol{I})^{-1}$$

where $\boldsymbol{S}_{(M)}$ is formed by removing the vector $\boldsymbol{s}_M$. Applying the matrix inversion lemma, we then have

$$\boldsymbol{B} = \boldsymbol{B}_{(M)} - \frac{\boldsymbol{B}_{(M)}\boldsymbol{s}_M\boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}^\dagger}{1 + \boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}\boldsymbol{s}_M}$$

which implies that

$$T_p(\boldsymbol{B}) = T_p\left(\boldsymbol{B}_{(M)}\right) - \frac{T_p\left(\boldsymbol{B}_{(M)}\boldsymbol{s}_M\boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}\right)}{1 + \boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}\boldsymbol{s}_M}.$$

Now, we partition $\boldsymbol{B}_{(M)}$ into M submatrices of size $MN \times N$

$$\boldsymbol{B}_{(M)} = \left(\boldsymbol{B}_{(M)}[0], \ldots, \boldsymbol{B}_{(M)}[p], \ldots, \boldsymbol{B}_{(M)}[M-1]\right)$$

where $\boldsymbol{B}_{(M)}[p]$ is formed by choosing the columns from $pN+1$ to $(p+1)N$. Since $\boldsymbol{B}_{(M)}$ is Hermitian, it can be shown that

$$T_p\left(\boldsymbol{B}_{(M)}\boldsymbol{s}_M\boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}\right) = \boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}[p]\boldsymbol{B}_{(M)}[p]^\dagger\boldsymbol{s}_M$$

and

$$T_p(\boldsymbol{B}) = T_p\left(\boldsymbol{B}_{(M)}\right) - \frac{\boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}[p]\boldsymbol{B}_{(M)}[p]^\dagger\boldsymbol{s}_M}{1 + \boldsymbol{s}_M^\dagger\boldsymbol{B}_{(M)}\boldsymbol{s}_M}. \tag{34}$$

The form of (34) is similar to that in the one-shot case except that the matrix $\boldsymbol{B}_{(M)}^2 = \boldsymbol{B}_{(1)}^2$ in the earlier case is now replaced by the matrix $\boldsymbol{B}_{(M)}[p]\boldsymbol{B}_{(M)}[p]^\dagger$. The only property of $\boldsymbol{B}_{(1)}^2$ required in the proof of Theorem 1 is the fact that its spectral norm is uniformly bounded for all N in obtaining (28) and (31). Since the matrices $\boldsymbol{B}_{(M)}[p]\boldsymbol{B}_{(M)}[p]^\dagger$ are positive definite, we immediately have

$$\begin{aligned}\rho\left(\boldsymbol{B}_{(M)}[p]\boldsymbol{B}_{(M)}[p]^\dagger\right) &\le \rho\left(\sum_{m=0}^{M-1}\boldsymbol{B}_{(M)}[m]\boldsymbol{B}_{(M)}[m]^\dagger\right) \\ &= \rho\left(\boldsymbol{B}_{(M)}^2\right) \le \frac{1}{\sigma^4}.\end{aligned}$$

Thus, the spectral norm of $\boldsymbol{B}_{(M)}[p]\boldsymbol{B}_{(M)}[p]^\dagger$ is uniformly bounded as well. The remaining steps in the proof of Theorem 1, *viz.* the definitions (25)–(27) and the techniques to bound them, can now be carried through. In particular, note that the application of Lemma 1 in (32) and (33) holds when $M > 1$ as well.

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