Approximate Lower Bound for the SNR *of Matched Filters*

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ABSTRACT: *For the simple binary detection problem of detecting a known signal in the presence of additive noise, the matched filter is well known to yield the highest output signal-to-noise ratio* (SNR). *When the detection is carried out in discrete time, selecting an optimal filter length for a specific detection problem is important. Bounds on the* SNR *of the matched filter can assist in this selection. Exact bounds on the* SNR *can be computed in terms of the eigenvalues of the noise covariance matrix, but these bounds can be difficult to compute. An approximate lower bound for the* SNR *has been suggested recently by Martinez and Thomas (see Ref.* (2), Franklin Inst. *Vol.* 321, *No.* 5, *pp.* 251–260, 1986). *A supplement to this bound which is* more accurate *for* small *values of filter length is discussed in this paper. Some examples which delineate a comparison between the two approximate bounds are presented.*

I. Introduction

A simple binary detection problem requires the detection of a known deterministic signal vector **s** in the presence of an additive zero-mean interference noise vector **n**. When the covariance matrix of the noise vector is known, an optimal detector known as the *matched filter* (MF) can be designed for this detection problem. The MF is optimal in the sense that it maximizes the signal-to-noise ratio (*SNR*) at its output (**1**).

For a given noise covariance matrix and for a fixed signal length, the signal shape can be chosen in such a way that the MF yields the highest *SNR*. For a given signal length, theoretically exact bounds on the *SNR* of the MF can be computed in terms of the eigenvalues of the noise covariance matrix, but this can be a tedious task especially when the signal lengths are large. Easily computed bounds are sought for, which will assist in choosing an appropriate signal length for a specific detection problem. Martinez and Thomas (**2**) have recently suggested a simple approximation to the lower bound for the *SNR* of the MF. This approximate lower bound converges to the theoretical value for large signal lengths but differs significantly from it for smaller signal lengths. We propose an easily computed lower bound on the *SNR* which is a good approximation at small signal lengths but diverges from the theoretical value at large signal lengths. Our bound should thus serve as a supplement to the bound suggested by Martinez and Thomas.

II. Background

The problem of detecting the known signal in the presence of additive noise can be viewed as a binary hypothesis testing problem. The decision is based on an observation vector $\mathbf{x}$, of length N, which is composed of noise $\mathbf{n}$ under the hypothesis $\mathbf{H}_0$ and a known signal $\mathbf{s}$ added to noise $\mathbf{n}$ under the alternative $\mathbf{H}_1$.

$$\begin{aligned} \mathbf{H}_0: &\quad \mathbf{x} = \mathbf{n} \\ \mathbf{H}_1: &\quad \mathbf{x} = \mathbf{s}+\mathbf{n}. \end{aligned} \tag{1}$$

We assume here that signal has unit energy, i.e.,

$$\mathbf{s}^T\mathbf{s} = 1. \tag{2}$$

If the noise covariance matrix $\mathbf{R}$ is known, an optimal filter (the MF) can be designed for this detection problem (**1**). The MF can be shown to have an output SNR which is given by

$$SNR_o = \mathbf{s}^T\mathbf{R}^{-1}\mathbf{s}. \tag{3}$$

For a given noise covariance matrix $\mathbf{R}$, and for signals of constant energy, the SNR_o of the MF can take on a range of values depending on the choice of signal shape. Bounds on the SNR_o of the MF can be found in terms of the eigenvalues of $\mathbf{R}$.

The $N \times N$ covariance matrix $\mathbf{R}$ is positive definite and Hermitian. Hence its eigenvalues λ_i are real and positive and can be ordered as given below:

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N. \tag{4}$$

Since $\mathbf{s}^T\mathbf{s} = 1$, we get from the Rayleigh quotient theorem (**3**) that

$$\frac{1}{\lambda_N} \leqslant \mathbf{s}^T\mathbf{R}^{-1}\mathbf{s} \leqslant \frac{1}{\lambda_1}, \tag{5}$$

i.e. that

$$\frac{1}{\lambda_N} \leqslant SNR_o \leqslant \frac{1}{\lambda_1}. \tag{6}$$

If we make the assumption that the noise is *stationary* with unit variance the SNR at the input of the MF is given by

$$SNR_i = \frac{1}{N}. \tag{7}$$

The improvement in SNR due to the MF is given by

$$SNR_{\mathrm{MF}} = \frac{SNR_o}{SNR_i} = N \cdot \mathbf{s}^T\mathbf{R}^{-1}\mathbf{s}. \tag{8}$$

Hence we get from Eq. (6) that

$$\frac{N}{\lambda_N} \leqslant SNR_{\mathrm{MF}} \leqslant \frac{N}{\lambda_1}. \tag{9}$$

The bounds in Eq. (9) are tight bounds. In fact, it is easily shown that the lower bound is met when the signal is chosen to be the eigenvector of $\mathbf{R}$ corresponding to eigenvalue λ_N, and the upper bound is met when the signal is chosen to be the eigenvector corresponding to eigenvalue λ_1. Even though these bounds look very promising, it is difficult to use them in choosing an appropriate filter length N since they require the knowledge of the eigenvalues of the noise covariance matrix $\mathbf{R}$. Martinez and Thomas (**2**) have suggested a looser but easier to compute lower bound for the SNR when the noise is stationary.

For stationary noise, the covariance matrix is Toeplitz, i.e. it has constant values along all its diagonals parallel to the main diagonal. If we denote the value in the ith diagonal by r_i, $i = -(N-1), \ldots, -1, 0, 1, \ldots, (N-1)$, an upper bound on the largest eigenvalue λ_N of $\mathbf{R}$ can be found (**2**) in terms of these values.

$$\lambda_N \leqslant \sum_{i=-\infty}^{\infty} |r_i|. \tag{10}$$

Thus using Eq. (9), we get a lower bound for SNR_{MF} as

$$SNR_{\mathrm{MF}} \geqslant \frac{N}{\sum_{i=-\infty}^{\infty} |r_i|}. \tag{11}$$

Since we shall be referring to these bounds often, we shall denote the tight lower bound of Eq. (9) by $\mathrm{LB_t}$ and the loose bound of Eq. (11) by $\mathrm{LB_l}$. It has been shown through some examples (**2**) that $\mathrm{LB_l}$ converges to $\mathrm{LB_t}$ for large values of N. But for small values of N there is a considerable difference between these two bounds. In this paper we introduce another easily computable lower bound on SNR_{MF} which is more accurate for small values of N and would thus serve as a supplement to $\mathrm{LB_l}$. We shall discuss this bound in the next section.

III. Supplementary Bound

A host of matrix inequalities, specifically relating to bounds on the eigenvalues of the matrix, can be formulated in terms of the trace of the matrix and the trace of its square (**4**). An upper bound on the largest eigenvalue λ_N of $\mathbf{R}$, written in terms of the trace of $\mathbf{R}$ and the trace of $\mathbf{R}^2$, is as follows:

$$\lambda_N \leqslant m + s\sqrt{N-1}, \tag{12}$$

where m and s are defined as

$$m = \frac{\mathrm{Tr}\{\mathbf{R}\}}{N},$$
$$s^2 = \frac{\mathrm{Tr}\{\mathbf{R}^2\}}{N} - m^2. \tag{13}$$

Here, $\mathrm{Tr}\{\cdot\}$ stands for the trace of the matrix within the brackets.

From Eq. (12), we get a lower bound on SNR_{MF} as shown below. We shall refer to this bound as LB_2:

$$SNR_{\mathrm{MF}} \geqslant \frac{N}{m+s\sqrt{N-1}}. \tag{14}$$

With the previously stated assumption that the noise is stationary with unit variance the expressions for m and s defined in Eq. (13) reduce to

$$\begin{aligned} m &= \frac{N}{N} = 1, \\ s^2 &= 2\sum_{i=1}^{N-1} r_i^2\left(1-\frac{i}{N}\right). \end{aligned} \tag{15}$$

From Eqs (14) and (15), we get

$$\mathrm{LB}_2 = \frac{N}{1+\sqrt{2(N-1)\sum_{i=1}^{N-1} r_i^2\left(1-\frac{i}{N}\right)}}. \tag{16}$$

It should be noted that the bound on λ_N given in Eq. (12) does not require $\mathbf{R}$ to be Toeplitz. Using this, one can derive a bound on the SNR of the filter for a general case when the noise is non-stationary. In that sense our bound is less restrictive than the bound LB_1.

IV. Some Examples

Martinez and Thomas (**2**) chose four noise autocorrelation functions to illustrate the accuracy of their lower bound. We have chosen the same four examples to delineate the domains where each of the bounds LB_1 and LB_2 is more accurate than the other. Before we describe these autocorrelation functions we introduce a quantity called the correlation length of the noise (l), which represents the time extent of the noise autocorrelation function.

The four noise autocorrelation functions are

the exponential:

$$r_i = \exp\left(-\frac{2|i|}{l}\right), \tag{17}$$

the triangular:

$$r_i = \begin{cases} 1-\frac{|i|}{l} & \text{if} \quad |i| \leqslant l \\ 0 & \text{if} \quad |i| > l \end{cases}, \tag{18}$$

the Gaussian:

$$r_i = \exp\left(-\frac{\pi i^2}{l^2}\right), \tag{19}$$

the hyperbolic secant:

$$r_i = \operatorname{sech}\left(\frac{\pi i}{l}\right). \tag{20}$$

The parameters of these four correlation functions were chosen so that for each of them, the quantity $\Sigma_{i=-\infty}^{\infty} |r_i|$ approximately equals l. Thus LB_l for each of these functions is given approximately by

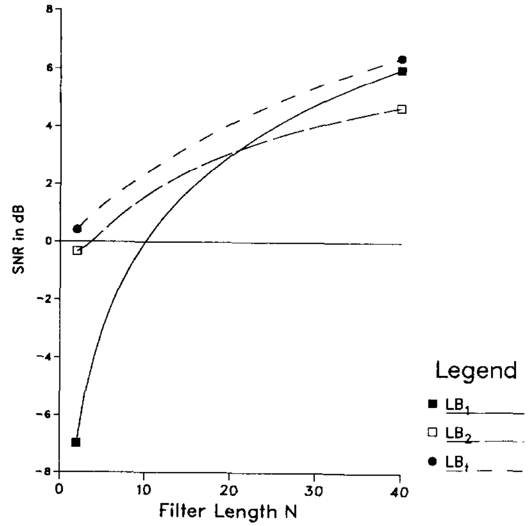
$$LB_l \approx \frac{N}{l}. \tag{21}$$

We computed LB_t, LB_l and LB_2 as a function of filter length N for the four examples given above with $l = 10$ (we chose $l = 10$ in order to compare with the results of Martinez and Thomas (**2**)). These bounds have been plotted in Figs 1(a)–1(d). As we can see, for large values of N, LB_l converges to LB_t. But for small N (i.e. $N < 20$) there can be as much as 6 dB difference between these two. LB_2 on the other hand is more accurate for small values of N. The graphs of LB_l and LB_2 intersect at $N \approx 15$. For values of N to the right of the intersection point the more accurate bound is LB_l, and for values to the left of this point LB_2 is more accurate. Hence, for a particular value of N the more accurate bound is the larger of LB_l and LB_2.

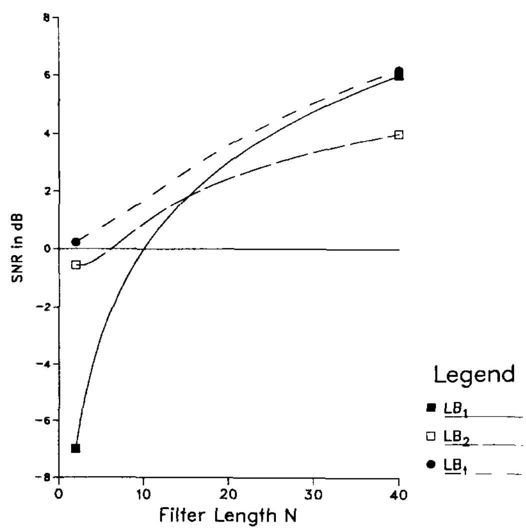
We also observed that the value of N at which LB_l and LB_2 intersect is a function of the correlation length l. Figures 2(a)–2(d) show plots of the three lower bounds for the exponential correlation function with three different values of l. From these plots, it can be seen that the value of N at the intersection is an approximately linear function of l. A similar behaviour was also observed (but not shown here) for the other three correlation functions. Thus we suggest the computation of both lower bounds in general. When the correlation length l is seen to be smaller than the filter length N, LB_l should be better than LB_2. On the other hand, if l is greater than N, we should compute and use LB_2.

V. Conclusions

Signal selection is important for the MF to perform optimally. The tools that can aid in signal length (or filter length) selection are bounds on the *SNR* of the filter which are computed as a function of the filter length. Since exact bounds on the *SNR* are difficult to obtain, looser bounds have to be used in signal length selection. We have derived an approximation to the lower bound of the *SNR* which is more accurate than an existing approximation, for small values of filter length. We suggest that for a specific detection problem both the loose lower bounds must be computed, and, for a particular value of filter length, the larger of the two computed bounds must be chosen as the more accurate bound.



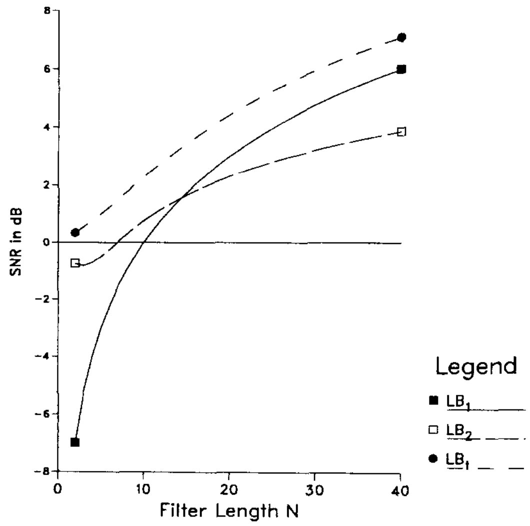
(a)



(b)

FIG. 1. LB_t, LB_1 and LB_2 for four different correlation functions with $l = 10$. (a) Exponential correlation. (b) Triangular correlation. (c) Gaussian correlation. (d) Hyperbolic secant correlation.

(c)



(d)

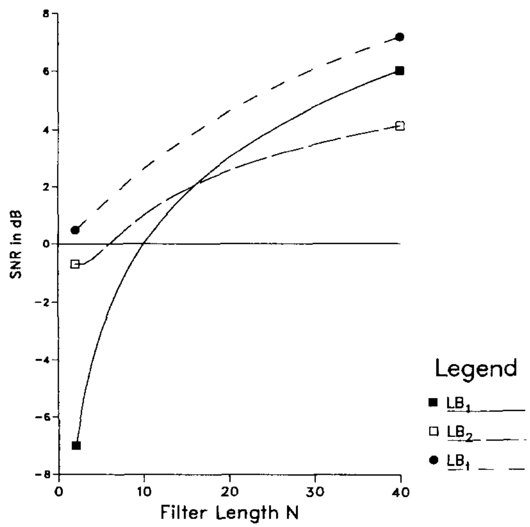


FIG. 1. (c) and (d).

(a)

(b)

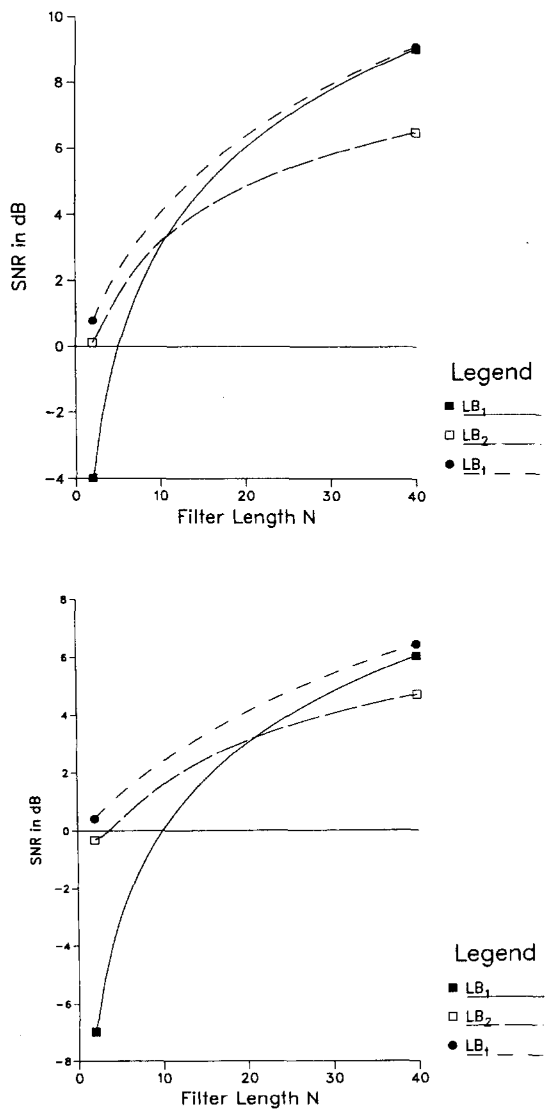


FIG. 2. LB_t, LB_l and LB_2 for the exponential correlation function for three different values of correlation length l. (a) $l = 5$. (b) $l = 10$. (c) $l = 20$.

(c)

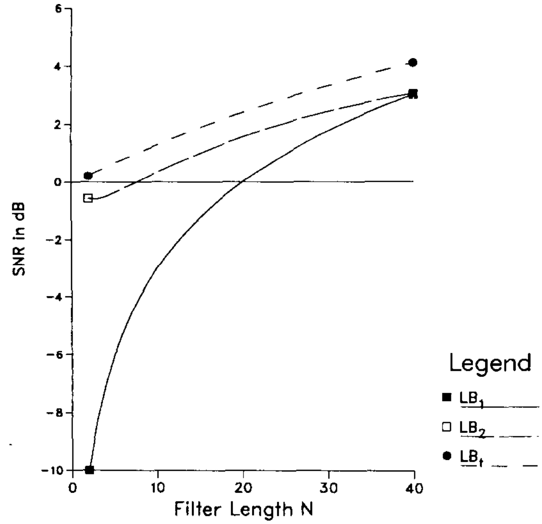


FIG. 2. (c).

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(4) H. Wolkowicz and G. P. H. Styan, "Bounds on eigenvalues using traces", *Linear Algebra and Appl.*, Vol. 29, No. 2, pp. 471–506, Feb. 1980.