ECE 559 Fall 2007 (1)

SOLUTIONS TO HW 4

1 (i) $$X_{MF} = \sum_{k=1}^{K} \cos(\phi_1 - \phi_k)\, A_k \sum_{i=-\infty}^{\infty} \mathfrak{z}_k^{(\lfloor i/N \rfloor)} c_{k,i}$$
$$\cdot \sum_{m=0}^{N-1} c_{1,m} \langle g_{T_c}(t - iT_c - \tau_k), g_{T_c}(t - mT_c) \rangle$$
$$+ w_1$$

where it is easily seen that $w_1 \sim N(0, \frac{N_0}{2})$

Separating out the $k=1$ term we get

$$X_{MF} = A_1 \mathfrak{z}_1^{(0)} \sum_{m=0}^{N-1} c_{1,m}^2 \langle g(t - mT_c), g(t - mT_c) \rangle$$
$$+ \sum_{k=2}^{K} \sum_{m=0}^{N-1} c_{1,m} A_k \sum_{i=-\infty}^{\infty} \mathfrak{z}_k^{(\lfloor i/N \rfloor)} c_{k,i} \cos(\phi_k - \phi_1) \cdot$$
$$\cdot \underbrace{\langle g(t - iT_c - \tau_k), g_{T_c}(t - mT_c) \rangle}_{= R_g(i - m + d_k)}$$

with $d_k = \tau_k / T_c$.

$$\Rightarrow X_{MF} = A_1 \mathfrak{z}_1^{(0)} + \sum_{k=2}^{K} \sum_{m=0}^{} c_{1,m}\, y_k[m] + W_1$$

(ii) $$E[X_k] = \sum_{m=0}^{M-1} E[c_{1,m} \cdot y_k[m]]$$

Since $\{c_{k,m}\}$, $\{\mathfrak{z}_k^{(i)}\}$ and $\{\phi_k\}$ are independent and $\{c_{1,m}\}$ and $\{c_{k,m}\}$ are independent for $k \neq 1$

$$E[x_k] = \sum_{m=0}^{N-1} \sum_{i=-\infty}^{\infty} A_k \, \overbrace{\cancel{E\left[z_k^{(\lfloor \frac{i}{N} \rfloor)}\right]}}^{=0} \, \overbrace{\cancel{E[\cos(\phi_k - \phi_1)]}}^{=0} \cdot$$
$$\cdot E[c_{1,m} \, c_{k,i}] \, E[R_g(i + d_k - m)]$$

$$\Rightarrow E[x_k] = 0$$

$$E[x_k^2 \,|\, d_k] = E\left[\sum_{m=0}^{N-1} \sum_{l=0}^{N-1} c_{1,m} \, c_{1,l} \, y_k[m] \, y_k[l] \,\Big|\, d_k \right]$$

For $k > 1$, $\{c_{1,m}, c_{1,l}\}$ are independent of $\{y_k[m], y_k[l]\}$

$$\Rightarrow E[x_k^2 \,|\, d_k] = \sum_{m=0}^{N-1} \sum_{l=0}^{N-1} \overbrace{E[c_{1,m} \, c_{1,l} \,|\, d_k]}^{=0,\, m \neq l} \cdot$$
$$E[y_k[m] \, y_k[l] \,|\, d_k]$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} E\left[(y_k[m])^2 \,|\, d_k\right]$$

$$E\left[(y_k[m])^2 \,|\, d_k\right] = E\Big[A_k^2 \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} z_k^{(\lfloor \frac{i}{N} \rfloor)} \, z_k^{(\lfloor \frac{j}{N} \rfloor)} \, c_{k,i}$$
$$c_{k,j} \cdot \cos^2(\phi_k - \phi_1) \cdot R_g(i + d_k - m)$$
$$\cdot R_g(j + d_k - m) \,\Big|\, d_k \Big]$$

$z_k^{(\lfloor \frac{i}{N} \rfloor)}$ and $z_k^{(\lfloor \frac{j}{N} \rfloor)}$ are independent for $i \neq j$

$$\Rightarrow E\left[(y_k[m])^2 \,|\, d_k\right] = \sum_{i=-\infty}^{\infty} \overbrace{E\left[\left(z_k^{(\lfloor \frac{i}{N} \rfloor)}\right)^2\right]}^{1} A_k^2$$
$$\cdot \underbrace{E\left[\cos^2(\phi_k - \phi_1)\right]}_{\frac{1}{2}}$$
$$\cdot E\left[R_g^2(i + d_k - m) \,|\, d_k\right]$$
$$\cdot E\left[c_{k,i}^2\right]$$

$$\Rightarrow E\left[X_k^2 \mid d_k\right] = \frac{A_k^2}{2N^2} \sum_{m=0}^{N-1} \sum_{i=-\infty}^{\infty} R_g^2(i+d_k-m)$$

$$E\left[X_k^2\right] = \frac{A_k^2}{2N^2} \sum_{m=0}^{N-1} \underbrace{\sum_{i=-\infty}^{\infty} \int_0^1 R_g^2(i+u-m)\,du}_{= \int_{-\infty}^{\infty} R_g^2(u)\,du \,=\, \sigma_g^2}$$

$$= \frac{A_k^2}{2N^2} N \sigma_g^2 = \frac{A_k^2}{2N} \sigma_g^2$$

$k=1$, can be handled separately but is not needed for part (iii)

(iii) $$X_{MF} = A_1 z_1^{(0)} + \underbrace{\sum_{k=2}^{K} X_k + W_1}_{\mathcal{N}(0,\sigma^2)}$$

$$\sigma^2 = \frac{N_0}{2} + \sum_{k=2}^{K} \frac{A_k^2 \sigma_p^2}{2N}$$

$$\Rightarrow P_b = Q\left(\frac{A_1}{\sigma}\right) = Q\left(\sqrt{\frac{2A_1^2}{N_0 + \sum_{k=2}^{K} \frac{A_k^2 \sigma_p^2}{N}}}\right)$$

$$\approx Q\left(\sqrt{\frac{2N}{(K-1)\sigma_p^2}}\right)$$ for $A_1 = A_2 \cdots = A_K$ and large K

(IV) For the rectangular pulse, it is easy to show that

$$\sigma_g^2 = 2\int_0^1 (1-u)^2\,du = \frac{2}{3}$$

and using Fourier Transforms, for the sinc pulse,

$$\sigma_g^2 = \int_{-\infty}^{\infty} \text{sinc}^2(u)\, du = 1$$

Thus, for a given P_b, we can accomodate a larger K with the rectangular chip pulse than the sinc pulse. But this increase in capacity is artificial since the rectangular pulse results in a larger bandwidth than the sinc pulse. In fact, as shown in reference [4] of the notes on CDMA, with bandwidth constraints, the capacity is maximized if the sinc pulse is used

(v) When the users are synchronous, $d_K = 0, \forall K$ and

$$E[X_K^2] = \frac{A_K^2}{2N^2} \sum_{m=0}^{N-1} \underbrace{\sum_{i=-\infty}^{\infty} R_g^2(i-m)}_{=\delta[i-m]}$$

$$= \frac{A_K^2}{2N^2} \sum_{m=0}^{N-1} 1 = \frac{A_K^2}{2N}$$

$$\Rightarrow P_b = Q\left(\sqrt{\frac{2A_1^2}{N_0 + \sum_{K=2} A_K^2/N}}\right) \rightarrow Q\left(\sqrt{\frac{2N}{(K-1)}}\right)$$

which is the same as P_b in asynchronous case with the sinc waveform This has been used to argue that asynchrony can increase the user capacity by a factor of $\frac{3}{2}$ if the rectangular pulse is used. Such an argument is obviously flawed.

2) $$SIR_k = \frac{E\left[\,|E[y_k|z_k]|^2\right]}{var(y_k|z_k)}$$

$$E[y_k|z_k] = A_k z_k \Rightarrow E\left[\,|E[y_k|z_k]|^2\right] = A_k^2$$

$$var(y_k|z_k) = var\Big(\sum_{l\neq k} A_l z_l \rho_{l,k} + w_k\Big)$$

$$= E\Big[\sum_{l\neq k} A_l^2 \rho_{l,k}^2\Big] + N_0$$

$$= \sum_{l\neq k} A_l^2\, E[\rho_{l,k}^2] + N_0$$

$$\rho_{l,k}^2 = \Big(\sum_{i=0}^{N-1} c_{l,i}\, c_{k,i}\Big)\Big(\sum_{j=0}^{N-1} c_{l,j}\, c_{k,j}\Big)$$

independent ~~for~~ $k\neq l$, $i\neq j$

Thus $$E[\rho_{l,k}^2] = \sum_{i=0}^{N} E[c_{l,i}^2\, c_{k,i}^2]$$

$$= \sum_{i=0}^{N} E[c_{l,i}^2]\, E[c_{k,i}^2]$$

$$= N\cdot\frac{1}{N}\cdot\frac{1}{N} = \frac{1}{N}$$

$$SIR_k = \frac{A_k^2}{\sum_{l\neq k} A_l^2\cdot\frac{1}{N} + N_0} = \frac{\frac{\mathcal{E}_{b,k}}{N_0}}{1+\frac{1}{N}\sum_{l\neq k}\frac{\mathcal{E}_{b,l}}{N_0}}$$

The BER expression is obvious from eq (41) in the notes.

3. From 2, we can conclude that the SIR per Walsh chip is given by

$$\gamma_c = \frac{\mathcal{E}_c/N_0}{\frac{K-1}{N}\frac{\mathcal{E}_c}{N_0}+1}, \quad \mathcal{E}_c = \text{energy per Walsh chip}$$

ν info bits map to M Walsh chips

$$\Rightarrow \quad \mathcal{E}_c = \frac{\nu \mathcal{E}_b}{M} \quad \text{and} \quad \gamma = \frac{M}{\nu}\gamma_c$$

Thus $$\gamma = \frac{M}{\nu}\gamma_c = \frac{\mathcal{E}_b/N_0}{\frac{K-1}{N}\frac{\nu}{M}\frac{\mathcal{E}_b}{N_0}+1}$$

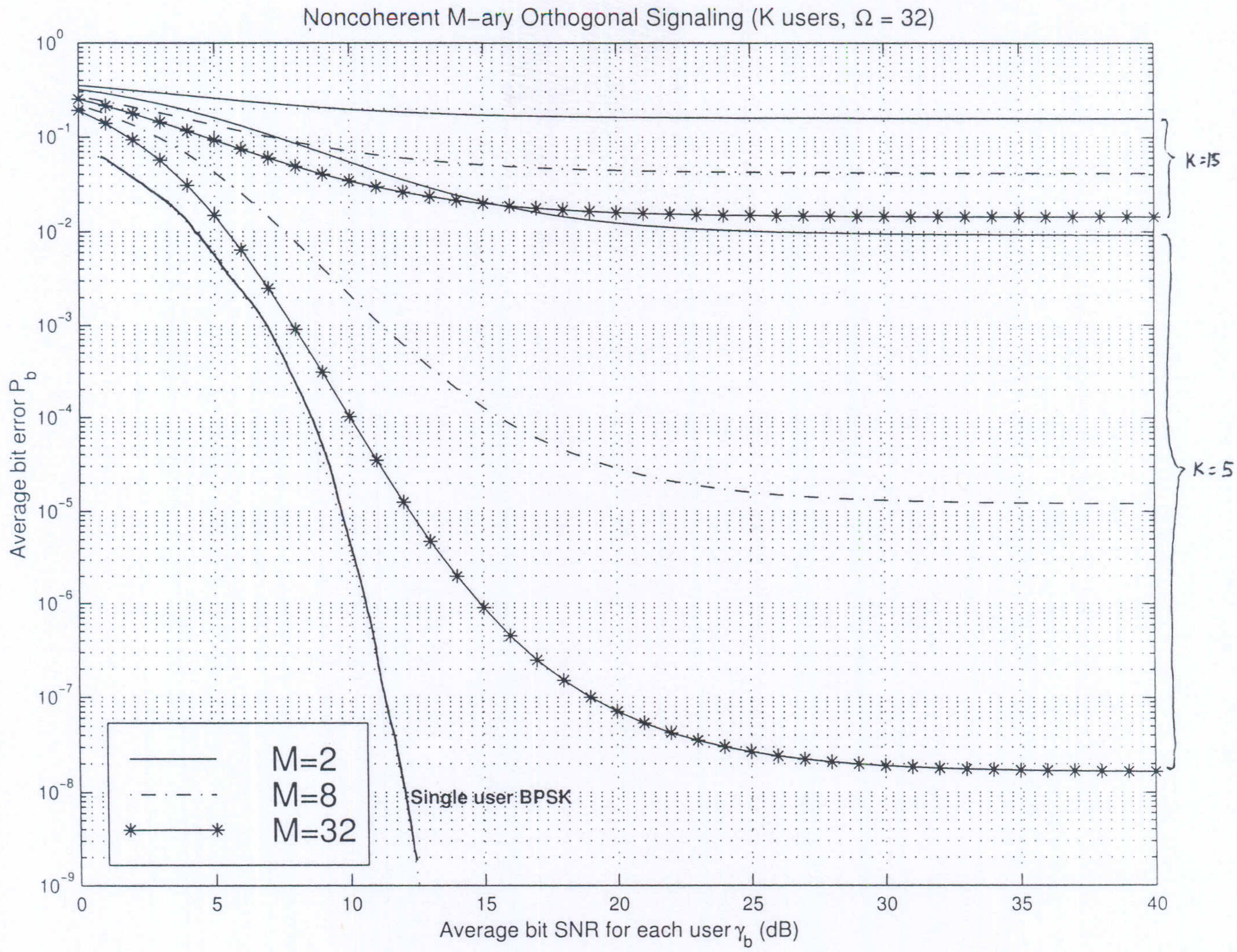
(ii) Orthogonal modulation results in a bandwidth expansion of $\frac{M}{\nu}$ and spreading by $N \Rightarrow \Omega = \frac{MN}{\nu}$

From problem 6 of HW 2,

$$P_b = \sum_{n=1}^{M-1} (-1)^{n+1}\binom{M-1}{n}\frac{1}{n+1}\exp\left[-\frac{n\nu\mathcal{E}_b}{(n+1)I_0}\right]$$

$$I_0 = \frac{(K-1)}{N}\frac{\nu}{M}\mathcal{E}_b + N_0$$

From the attached figure it is clear that larger values of M are better, i.e. it is best to put all of the bandwidth expansion into orthogonal modulation (or coding). The coding spreading tradeoff favors coding



(iii) For orthogonal sequences, the coding spreading tradeoff will favor spreading since we can obtain SU performance in this fashion. For MUD, again the tradeoff would favor spreading, although we ~~may~~ should expect less spreading than in the orthogonal sequence case.

4. $$p(\underline{y}/\underline{z}) = \frac{1}{\pi^K |\det(N_0 R)|} e^{-(\underline{y}-RA\underline{z})^{\dagger}(N_0 R)^{-1}(\underline{y}-RA\underline{z})}$$

$$\hat{\underline{z}}_{ML} = \arg\max_{\underline{z}} \; 2\,\mathrm{Re}[\underline{y}^{\dagger} A \underline{z}] - \underline{z}^{\dagger} ARA \underline{z}$$

For multistage detector,

$$\hat{z}_k(m+1) = \arg\max_{z_k} \; 2\,\mathrm{Re}[\underline{y}^{\dagger} A \underline{z}] - \underline{z}^{\dagger} ARA \underline{z}$$

with $z_\ell = \hat{z}_\ell(m)$ for $\ell \neq k$

$$= \arg\max_{z_k} \left(2\,\mathrm{Re}\Big[\sum_{\ell} y_\ell^* A_\ell z_\ell\Big] - \underline{z}^{\dagger} ARA \underline{z} \right)$$

Drop all terms without z_k or z_k^* $\Rightarrow$

$$\hat{z}_k(m+1) = \arg\max_{z_k} \; 2\,\mathrm{Re}[y_k^* A_k z_k] - A_k^2 |z_k|^2 - 2\,\mathrm{Re}\Big[\sum_{\ell \neq k} A_\ell z_\ell(m) z_k^* A_k^* \rho_{\ell,k}\Big]$$

$$= \arg\max_{z_k} \; \mathrm{Re}\Big[y_k^* A_k z_k - \sum_{\ell \neq k} A_\ell A_k \rho_{\ell,k} z_\ell(m) z_k^*\Big] - \frac{A_k^2 |z_k|^2}{2}$$

When z_k is binary,

$$\hat{z}_k(m+1) = \arg\max_{z_k} \; A_k z_k \,\mathrm{Re}\Big[y_k^* - \sum_{\ell \neq k} \rho_{\ell k} \hat{z}_\ell(m) A_\ell\Big]$$

$$= \mathrm{sgn}\Big[\mathrm{Re}\Big[y_k - \sum_{\ell \neq k} \rho_{\ell,k} A_\ell \hat{z}_\ell(m)\Big]\Big]$$

5. Optimality of decorrelator (assume R, $\underline{y}$ are real valued)

$$\hat{\underline{z}}_{opt} = \arg\max_{\underline{z}} \max_{A_k > 0} \left[2\,\underline{y}^T A \underline{z} - \underline{z}^T A R A \underline{z} \right]$$

Let $A\underline{z} = \underline{x}$. Then $\underline{z} = \mathrm{sgn}(\underline{x})$, since $A_k > 0$. The maximization above can be recast as

$$\max_{\underline{x} \in \mathbb{R}^K} \left(2\,\underline{y}^T \underline{x} - \underline{x}^T R \underline{x} \right)$$

The value of $\underline{x}$ that maximizes this function is

$$\underline{x}^* = R^{-1} \underline{y}$$

Why? If $\underline{x} = \underline{x}^* + \underline{c} = R^{-1}\underline{y} + \underline{c}$

Then $$2\,\underline{y}^T \underline{x} - \underline{x}^T R \underline{x} = 2\,\underline{y}^T R^{-1} \underline{y} + 2\,\underline{y}^T \underline{c} - 2\,\underline{y}^T \underline{c} - \underline{c}^T R^{-1} \underline{c} - \underline{y}^T R^{-1} \underline{y}$$

$$= \underline{y}^T R^{-1} \underline{y} - \underline{c}^T R^{-1} \underline{c}$$

$\Rightarrow$ $\underline{c}$ only decreases the function

$$\hat{\underline{z}}_{opt} = \mathrm{sgn}(\underline{x}^*) = \mathrm{sgn}(R^{-1}\underline{y})$$