

Controlled Sensing for Multihypothesis Testing

Sirin Nitinawarat, *Member, IEEE*, George K. Atia, *Member, IEEE*, and Venugopal V. Veeravalli, *Fellow, IEEE*

Abstract—The problem of multiple hypothesis testing with observation control is considered in both fixed sample size and sequential settings. In the fixed sample size setting, for binary hypothesis testing, the optimal exponent for the maximal error probability corresponds to the maximum Chernoff information over the choice of controls, and a pure stationary open-loop control policy is asymptotically optimal within the larger class of all causal control policies. For multihypothesis testing in the fixed sample size setting, lower and upper bounds on the optimal error exponent are derived. It is also shown through an example with three hypotheses that the optimal causal control policy can be strictly better than the optimal open-loop control policy. In the sequential setting, a test based on earlier work by Chernoff for binary hypothesis testing, is shown to be first-order asymptotically optimal for multihypothesis testing in a strong sense, using the notion of decision making risk in place of the overall probability of error. Another test is also designed to meet hard risk constraints while retaining asymptotic optimality. The role of past information and randomization in designing optimal control policies is discussed.

Index Terms—Chernoff information, controlled sensing, design of experiments, detection and estimation theory, error exponent, hypothesis testing, Markov decision process.

I. INTRODUCTION

THE topic of controlled sensing for inference deals primarily with adaptively managing and controlling multiple degrees of freedom in an information-gathering system, ranging from the sensing modality to the physical control of sensors, to achieve a given inference task. Unlike in traditional control systems, where the control primarily affects the evolution of the state, in controlled sensing, the control affects only the observations. The goal is for the decision-maker to infer the state accurately by shaping the quality of observations.

Some applications of controlled sensing include, but are by no means limited to, target detection, tracking and classification

(see, e.g., [1], [2]). Controlled sensing policies were also developed for landmine and underwater mine classification in [3]. In the domain of sequential clinical trials, controlled sensing has been used to planning of medical trials under an ethical injunction against unnecessary continuance of inferior treatments [4]. Dynamic sensor selection and scheduling policies were also developed for tracking and target localization in [5], [6].

In this paper, we focus on the basic inference problem of hypothesis testing, and our goal is to find an asymptotically optimal joint-design of a control policy and a decision rule (in addition to a stopping rule for the sequential setting) to decide among the various hypotheses [7]. In particular, we consider a Markovian model for the simple hypothesis testing of multiple hypotheses with observation control. Prior to making a decision about the hypothesis, the decision-maker can choose among different actions to shape the quality of the observations. We consider both the fixed sample size and sequential settings of this problem. In the latter setting, the controller can adaptively choose to stop taking observations, and the sequential test is fully described by a control policy, a stopping rule and a final decision rule.

A. Relationship to Prior Work

We begin by discussing prior work in the fixed sample size setting. Tsitsiklis [8] considered the problem of quantizing independent observations at geographically separated sensors for multiple hypothesis testing. The number of sensors, which is taken to infinity in [8], can be considered to be equivalent to the sample size in our controlled sensing problem. Therefore, the quantization rules can be considered to be special cases of control actions that can affect the observations at the output of the various sensors. However, the control actions in the controlled sensing problem are much more general. Furthermore, the observation control policy in [8] is effectively an open-loop control policy. In contrast, our main focus in this paper is on temporal observation control in which the control at each time can be influenced by the past observations.

In the fixed sample size setting, the block channel coding problem with feedback and with a *fixed number of messages* studied by Berlekamp [9] can also be considered to be a special case of the controlled sensing problem. This is because, in the coding problem, the controller (encoder) has access to the hypothesis (message), whereas in our controlled sensing problem the controller is not assumed to have access to the hypothesis and is therefore more challenging.

The controlled sensing problem is also more general than the multi-channel identification problem treated by Mitran and Kavčić [10], in which there is a *finite* constraint on the number of past channel outputs available to the input signal selector at each time. In contrast, the causal control policies considered herein can depend on the entire past observations, the number of which

Manuscript received May 03, 2012; revised November 10, 2012, March 13, 2013, and April 15, 2013; accepted March 13, 2013. Date of publication May 03, 2013; date of current version September 18, 2013. This work was supported by the Air Force Office of Scientific Research (AFOSR) under the Grant FA9550-10-1-0458 through the University of Illinois at Urbana-Champaign, by the National Science Foundation through subcontract 147755 at the University of Illinois from prime award HDTRA1-10-1-0086, and by the National Science Foundation under Grant NSF CCF 11-11342. The material in this paper was presented in part at the IEEE International Conference on Acoustic, Speech and Signal Processing Kyoto, Japan March 2012, and at the IEEE International Symposium on Information Theory Cambridge, USA July 2012. Recommended by Associate Editor L. Schenato.

S. Nitinawarat and V. V. Veeravalli are with the Department of Electrical and Computer Engineering and the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801 USA (e-mail: nitinawa@illinois.edu, vvv@illinois.edu).

G. K. Atia is with the Department of Electrical Engineering and Computer Science, University of Central Florida, Orlando, FL 32816-2362 USA (e-mail: george.atia@ucf.edu).

Digital Object Identifier 10.1109/TAC.2013.2261188

becomes *unbounded* as the horizon approaches infinity. In related work, Hayashi [11] considered the adaptive discrimination of channels with unbounded memory, but for only *two* channels, i.e., two hypotheses.

In Section III, we first present a characterization for the optimal error exponent for binary hypothesis testing with a fixed sample size showing that a pure stationary open-loop control, where the control value at each time is fixed and does not depend on past measurements and past controls, achieves the optimal error exponent among the class of causal controls. In fact, this result is in agreement with that of Hayashi on discrimination of two channels [11] (see also Footnote 2). Then, for general multiple hypothesis testing with a fixed sample size, we derive a characterization for the optimal error exponent achievable by open-loop control. With more than two hypotheses, the characterization for the optimal error exponent achievable by causal control (which can be a function of past measurements and past controls) is a more difficult problem. Nevertheless, we show through a concrete example with *only* three hypotheses that the optimal causal control policy can be strictly better than the optimal open-loop control policy. We also derive general lower and upper bounds for the optimal error exponent achievable by causal control.

We now discuss related work in the sequential setting. The problem of sequential hypothesis testing *without control* was introduced by Wald [12], [13] and studied in detail for the binary hypothesis case. In this work, the optimal expected values of the stopping time were characterized subject to constraints on the probabilities of error under each hypothesis. It was shown that the Sequential Probability Ratio Test (SPRT) is optimal, i.e., among all tests with the same power, the SPRT requires on average the fewest number of observations. An extension to the multihypothesis case was considered in [14] where the authors proposed a Multihypothesis SPRT (or MSPRT) which was later shown to satisfy certain asymptotic optimality conditions [15]–[17].

The problem of sequential binary composite hypothesis testing *with observation control* was considered by Chernoff [18] and an asymptotically optimal sequential test was presented. While Wald's SPRT is optimal in the sense that it minimizes the expected values of the stopping time among all tests for which the probabilities of error do not exceed predefined thresholds [13], a weaker notion of optimality is adopted in [18]. Specifically, the proposed test is shown to achieve optimal expected values of the stopping time subject to the constraints of *vanishing* probabilities of error under each hypothesis. The sequential test with causal control proposed by Chernoff can only be proven to be asymptotically optimal under a set of positivity constraints on the Kullback-Leibler distances as defined in (12). Bessler [19] generalized Chernoff's work to general multiple hypothesis testing but also imposed the same type of assumption on the model.¹

Burnašev [20] considered the problem of sequential discrimination of multiple hypotheses with control of observations under a different information structure. It is important to note

that the controlled sensing problem that we consider is fundamentally different from Burnašev's problem. Unlike [20], where the control actions are functions of the underlying hypothesis, in [18] and the setting we consider herein the control actions cannot be functions of the unknown hypothesis. In that sense, the problem considered in [20] has a simpler structure since the controller knows the underlying hypothesis. This knowledge simplifies both the optimization of control policies as well as their performance analysis. When the hypothesis is unknown to the controller, as in the controlled sensing problem considered herein, the controller has to base its control actions on estimates of the unknown hypothesis.

A Bayesian version of this sequential problem (with observation control) was considered by the authors in [21] in the non-asymptotic regime. Since the optimal policy is generally difficult to characterize, certain conditions (Blackwell ordering [22]) were identified under which the optimal control is an open-loop control. The main focus of [21], [23]–[25] has been on trying to solve the underlying dynamic program and finding the structure of optimal solutions, a task that is only possible in some special cases. In contrast, our work mostly focuses on performance analysis and on establishing the asymptotic optimality of proposed control policies.

In Section IV, we extend the results in [18], [19] in several directions. First, we show that the sequential test in [18], [19] is asymptotically optimal in a *strong* sense, using the notion of frequentist risks in place of the probability of error. Second, we dispense with the positivity assumption on the Kullback-Leibler divergences used in [18], [19], by constructing a modification to Chernoff's test that does not require this assumption. Third, we construct a further modification to the test that meets hard constraints on the frequentist risks, while retaining asymptotic optimality.

B. Paper Outline

The remainder of the paper is organized as follows. In Section II, we specify the general notations and assumptions that will be adopted throughout the paper. Our problem formulations and results for the fixed sample size setting and the sequential setting, together with a summary of our contributions in each case, are given in Sections III and IV, respectively; An example is provided in Section V. A discussion is provided in Section VI, and conclusions are given in Section VII.

II. PRELIMINARIES

Throughout the paper, random variables are denoted by capital letters and their realizations are denoted by the corresponding lower-case letters.

Consider hypothesis testing with M hypotheses, with the set of hypotheses denoted by $\mathcal{M} \triangleq \{0, \dots, M-1\}$. At each time step, the observation takes values in \mathcal{Y} and the control takes values in \mathcal{U} . We assume that the control alphabet \mathcal{U} is *finite*. The observation alphabet \mathcal{Y} is a measurable space; it can be either continuous, i.e., a finite-dimensional Euclidean space, or discrete. Under each hypothesis $i \in \mathcal{M}$, and at each time k , conditioning on the event that the current control u_k has value u ,

¹We would like to thank an anonymous reviewer for pointing us to the generalization of Chernoff's test to the $M > 2$ case in Bessler's dissertation.

the current observation Y_k is assumed to be conditionally independent of past observations and past controls $(y^{k-1}, u^{k-1}) \triangleq ((y_1, \dots, y_{k-1}), (u_1, \dots, u_{k-1}))$. We refer to this (conditionally) memoryless assumption as the stationary Markovity assumption.

The following technical assumptions are made throughout the paper. First, for every $u \in \mathcal{U}$, we assume that the distributions of the observations under each hypothesis $i \in \mathcal{M}$ are absolutely continuous with respect to a common distribution μ_u on \mathcal{Y} . Consequently, for every $u \in \mathcal{U}$ and every $i \in \mathcal{M}$, there exists a probability density function (pdf)/probability mass function (pmf) p_i^u such that for every measurable set $A \subseteq \mathcal{Y}$,

$$\mathbb{P}_i^u\{Y \in A\} = \int_A p_i^u(y) d\mu_u(y), \quad u \in \mathcal{U}, \quad (1)$$

where the notation \mathbb{P}_i^u denotes the probability measure with respect to the distribution p_i^u . Second, we also assume that for every $u \in \mathcal{U}$ and every pair $i, j \in \mathcal{M}$, $i \neq j$,

$$\mathbb{E}_i^u \left[\left(\log \left(\frac{p_i^u(Y)}{p_j^u(Y)} \right) \right)^2 \right] < \infty, \quad (2)$$

where the notation \mathbb{E}_i^u denotes an expectation with respect to p_i^u . Note that it follows from (2) that for every $u \in \mathcal{U}$ and every pair $i, j \in \mathcal{M}$, $i \neq j$, p_i^u is absolutely continuous with respect to p_j^u . However, for $u, u' \in \mathcal{U}$, $u \neq u'$, and $i, j \in \mathcal{M}$, p_i^u need not be absolutely continuous with respect to $p_j^{u'}$. For a finite \mathcal{Y} , the combination of (2) and the first assumption is tantamount to the condition that all pmfs in the collection $\{p_i^u\}_{i \in \mathcal{M}}$ have the same support. However, the support could be different for different values of u .

III. FIXED SAMPLE SIZE SETTING

In this section, we first consider the setting wherein the sample size is fixed a priori, i.e., it does not depend on specific realizations of the observations and controls.

We consider two classes of control policies based on two information patterns. The first is the open-loop control policy where the (possibly randomized) control sequence (U_1, \dots, U_n) is assumed to be independent of the observations (Y_1, \dots, Y_n) . The second is the causal control policy where at each time k , the control U_k can be any (possibly randomized) function of past observations and past controls, i.e., U_k , $k = 2, 3, \dots, n$, is described by an arbitrary conditional pmf $q_k(u_k | y^{k-1}, u^{k-1})$, and U_1 is distributed according to a pmf $q_1(u_1)$. If all these (conditional) pmfs are point-mass distributions, i.e., the current control is a deterministic function of past observations and past controls, then the resulting policy is a *pure* control policy. Under the aforementioned stationary Markovity assumption, the joint probability distribution function of (Y^n, U^n) under each hypothesis i , denoted by $p_i(y^n, u^n)$, can be written as

$$p_i(y^n, u^n) \triangleq q_1(u_1) \prod_{k=1}^n p_i^{u_k}(y_k) \prod_{k=2}^n q_k(u_k | y^{k-1}, u^{k-1}). \quad (3)$$

For open-loop control, $q_k(u_k | y^{k-1}, u^{k-1})$ is (conditionally) independent of y^{k-1} ; hence,

$$\begin{aligned} p_i(y^n, u^n) &= \left(\prod_{k=1}^n p_i^{u_k}(y_k) \right) \left(q_1(u_1) \prod_{k=2}^n q_k(u_k | u^{k-1}) \right) \\ &= \left(\prod_{k=1}^n p_i^{u_k}(y_k) \right) q(u^n). \end{aligned} \quad (4)$$

After n observations, a decision is made about the hypothesis according to the rule $\delta : \mathcal{Y}^n \times \mathcal{U}^n \rightarrow \mathcal{M}$ with maximal error probability: $e(\{q_k\}_{k=1}^n, \{p_i^u\}_{i \in \mathcal{M}}^{u \in \mathcal{U}}, \delta) \triangleq \max_{i \in \mathcal{M}} \mathbb{P}_i\{\delta \neq i\}$. Note that for a pure control policy, u^n is either a fixed sequence (pure open-loop control) or a deterministic function of the observations y^n (pure causal control). Consequently, when a pure control policy is adopted, it suffices to consider a decision rule that is a function only of the observations, i.e., $\delta(y^n, u^n) = \delta(y^n)$. The combination of a control policy and a decision rule will be referred to as a *test*. The asymptotic quantities of interest will be the largest exponent of the maximal error probability achievable by open-loop control, denoted by β_{OL} , and by causal control, denoted by β_C , respectively. In particular,

$$\begin{aligned} \beta_{OL} &\triangleq \overline{\lim}_n \sup_{\delta, q(u^n)} -\frac{1}{n} \log \left(e \left(q(u^n), \{p_i^u\}_{i \in \mathcal{M}}^{u \in \mathcal{U}}, \delta \right) \right); \\ \beta_C &\triangleq \overline{\lim}_n \sup_{\substack{\delta, q_1(u_1) \\ \{q_k(u_k | y^{k-1}, u^{k-1})\}_{k=2}^n}} -\frac{1}{n} \log \left(e \left(\{q_k\}_{k=1}^n, \{p_i^u\}_{i \in \mathcal{M}}^{u \in \mathcal{U}}, \delta \right) \right). \end{aligned}$$

It follows immediately from these definitions that $\beta_{OL} \leq \beta_C$, as the information pattern associated with causal control is more informative than that associated with open-loop control. We also seek to characterize the optimal control policies that achieve the optimal error exponents.

Note that because the number of hypotheses is fixed, we can consider a Bayesian probability of error (with respect to any prior probability distribution of the hypothesis) instead of the maximal one in the definitions of the optimal error exponents without changing their optimal values.

Before moving on to the technical part, we first summarize our contributions in this section.

- We derive a characterization for the optimal error exponent achievable by open-loop control for general multiple hypothesis testing with a fixed sample size (see Remark 3 explaining the connection between this result and previous work [10]).
- We propose a test for general multiple hypothesis testing with a fixed sample size using a causal control policy that chooses the control value based on a suitable Chernoff information. We also derive general lower and upper bounds for the optimal error exponent achievable by causal control that holds for any number of hypotheses, and illustrate through a canonical example with *only three* hypotheses that causal control can outperform open-loop control.

A. Case of Binary Hypothesis Testing ($M = 2$)

For p_1 and p_2 that are pdfs/pmfs on \mathcal{Y} with respect to a common distribution λ , the Kullback-Leibler (KL) distance of p_1 and p_2 , denoted by $D(p_1\|p_2)$, is defined as

$$D(p_1\|p_2) \triangleq \int_{\mathcal{Y}} p_1(y) \log \left(\frac{p_1(y)}{p_2(y)} \right) d\lambda(y).$$

We start with the following characterizations for the largest error exponents achievable by open-loop control and by causal control in the case of binary hypothesis testing.

For any $u \in \mathcal{U}$ and any $s \in [0, 1]$, consider the following pdf/pmf

$$b_s^u(y) \triangleq \frac{p_0^u(y)^s p_1^u(y)^{1-s}}{\int_{\mathcal{Y}} p_0^u(\bar{y})^s p_1^u(\bar{y})^{1-s} d\mu_u(\bar{y})}, \quad (5)$$

and also let

$$s^*(u) \triangleq \arg \max_{s \in [0,1]} -\log \left(\int_{\mathcal{Y}} p_0^u(y)^s p_1^u(y)^{1-s} d\mu_u(y) \right).$$

Proposition 1: For $M = 2$, it holds that²

$$\beta_{OL} = \beta_C$$

$$= \max_{u \in \mathcal{U}} \max_{s \in [0,1]} -\log \left(\int_{\mathcal{Y}} p_0^u(y)^s p_1^u(y)^{1-s} d\mu_u(y) \right) \quad (6)$$

$$= \max_{u \in \mathcal{U}} D(b_{s^*(u)}^u \| p_0^u) = \max_{u \in \mathcal{U}} D(b_{s^*(u)}^u \| p_1^u). \quad (7)$$

Remark 1: For each fixed $u \in \mathcal{U}$, the quantity

$$C(p_0^u, p_1^u) \triangleq \max_{s \in [0,1]} -\log \left(\int_{\mathcal{Y}} p_0^u(y)^s p_1^u(y)^{1-s} d\mu_u(y) \right) \quad (8)$$

is called the ‘‘Chernoff information’’ of p_0^u and p_1^u . Consequently, Proposition 1 (cf. (6)) states that the optimal error exponent is the maximum Chernoff information over the choice of controls.

Remark 2: It follows from Proposition 1 and the result on the Chernoff information for i.i.d. observations that the above optimal error exponent is achievable by a pure stationary open-loop control sequence in which, for every $k = 1, \dots, n$, $u_k = u^*$, where u^* is the maximizer associated with the right-side of (6) (or, identically, with the two (maximizing) optimization problems in (7)). In particular, *information from the past and randomization are superfluous for attaining the best error exponent for fixed sample size binary hypothesis testing.*

B. Case of Multiple Hypothesis Testing ($M > 2$)

1) *Open-Loop Control:* Our first theorem pertains to the situation with open-loop control.

²Although this result is mathematically equivalent to [11, Theorem 1] on discrimination of two channels, we point out here that the ‘‘discrimination’’ problem is motivated by the channel coding problem (see [20]), in which the controller can be considered to know the true hypothesis.

Theorem 1: For $M > 2$, it holds that

$$\beta_{OL} = \max_{q(u)} \min_{i \neq j} \max_{s \in [0,1]} -\sum_u q(u) \log \left(\int_{\mathcal{Y}} p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right), \quad (9)$$

where the left-most maximization is over all pmfs q on \mathcal{U} and the minimization is over all pairs of hypotheses i, j , $i \neq j$. Furthermore, β_{OL} is achievable by pure (non-randomized) control.

Remark 3: For finite observation alphabets, β_{OL} in (9) can be shown to be equal to the alternative formula derived in [10, Theorem 5]. However, our formula in (9) is simpler than that in [10, Theorem 5] because it involves maximization over a single real-valued spurious parameter s instead of minimization over a conditional distribution as in [10, Theorem 5]. More importantly, our result applies also to general observation alphabets not just the finite ones.

2) *Causal Control:* A natural question that arises now is whether causal control can yield a larger error exponent than open-loop control when $M > 2$. The answer will be shown to be affirmative even for $M = 3$. To this end, we now propose a test with *pure* causal control (we show in Theorem 2 below that pure causal control does achieve the optimal error exponent).

Our test admits the following recursive description and is based on the use of the posterior distribution of the hypothesis as a sufficient statistic. Having obtained the first k observations y^k , we find the maximum likelihood (ML) estimate of the hypothesis, denoted by $\hat{i}_k(y^k) = \arg \max_{i \in \mathcal{M}} p_i(y^k)$.³ We adopt a pure control policy wherein $u_{k+1} \in \mathcal{U}$ is selected as

$$u_{k+1} = u_{k+1}(\hat{i}_k) = \arg \max_{u \in \mathcal{U}} \min_{j \in \mathcal{M} \setminus \{\hat{i}_k\}} C(p_{\hat{i}_k}^u, p_j^u), \quad (10)$$

where $C(p_{\hat{i}_k}^u, p_j^u)$ is the Chernoff information of $p_{\hat{i}_k}^u$ and p_j^u defined in (8). Lastly, at the final time n , the decision rule is specified as $\delta(y^n, u^n) = \hat{i}_n$. *The proposed test follows the celebrated separation principle between estimation and control; while estimating the ML hypothesis is carried out online, the control is chosen based on a stationary deterministic mapping from the space of posterior distributions to the control space, and hence, the mapping can be fully specified offline.* It will be shown in Section V that for the special example with *only* three hypotheses, this proposed test is superior to the best open-loop control. In general, we still do not know the structure of the optimal causal control, and characterizing the optimal error exponent for causal control is a hard problem even for $M = 3$. Nevertheless, we derive precise bounds on the optimal error exponent that are applicable for any $M > 2$. Note that the optimal error exponent achievable by open-loop control as characterized in Theorem 1 already serves as a lower bound for the optimal error exponent achievable by causal control. We also derive a new lower bound and an upper bound for the optimal error exponent for causal control. These bounds are stated in Theorem 2 for the fixed sample size setting with $M > 2$. Although the

³In case of ties, we pick, say, the hypothesis with the least numerical value.

lower bound of Theorem 2 for β_C holds only for a *finite* observation alphabet \mathcal{Y} , the upper bound in Theorem 2 and all the previous results are valid for an arbitrary \mathcal{Y} (subject to assumptions (1) and (2) in Section II). As mentioned in Section II, for a finite \mathcal{Y} , we assume that for every $u \in \mathcal{U}$, the collection of pmfs $\{p_i^u\}_{i \in \mathcal{M}}$ have the same support.

For any pmf ν on \mathcal{M} , any $u \in \mathcal{U}$, let $\nu \circ p^u(\cdot)$ denote the pmf/pdf (on \mathcal{Y}) $\sum_i \nu(i) p_i^u(y)$.

Theorem 2: For every finite \mathcal{Y} and every $M > 2$, it holds that

$$\sup_{\eta > 0} -\log \left(\sup_{\nu} \min_u \max_i \left(\sum_y p_i^u(y) e^{\frac{\eta [\nu \circ p^u(y) - p_i^u(y)]}{(1-\nu(i))}} \right) \right) \leq \beta_C \leq \min_{i \neq j} \max_u \max_{s \in [0,1]} -\log \left(\sum_y p_i^u(y)^s p_j^u(y)^{1-s} \right), \quad (11)$$

where the outer supremum for the argument of $-\log$ in the lower bound is over pmfs ν on \mathcal{M} that are not point-mass distributions and the outer minimization for the upper bound in (11) is over all pairs of hypotheses i, j , $i \neq j$. Furthermore, as for β_{OL} , the exponent β_C is also achievable by pure control without any randomization.

Remark 4: The optimization problem for the lower bound in (11) can be handled off-line. In the example below, we show that the value of this lower bound is strictly larger than β_{OL} .

IV. SEQUENTIAL SETTING

In the previous section, we considered tests with a fixed sample size. In this section, we consider a different setting in which the controller can adaptively decide, based on the realizations of past observations and past controls, whether to continue collecting new observations, thereby deferring making a final decision about the hypothesis until later time, or to stop taking observations and make the final decision. In this setting, the goal is to design a *sequential* test to achieve the optimal tradeoff between reliability, in terms of probability of error, and delay or cost, in terms of the expected sample size needed for decision making. Unlike in the fixed sample size setting in which the asymptotic analysis of tests with open-loop control is easier than that of tests with causal control, in the sequential setting, the contrary situation seems to hold. In particular, as we show below, the adoption of randomized *causal* control in the sequential setting enables the simultaneous minimization of the expected sample sizes under the M hypotheses as the error probability vanishes. In contrast, *open-loop* control does not enable such simultaneous minimization, and therefore the characterization of the associated tradeoff is more difficult and remains open. This is why we only consider causal control in the sequential setting.

We now summarize our contributions in this setting.

- The existing sequential test originally proposed by Chernoff [18] for binary composite hypothesis testing, and extended to the multihypothesis setting by Bessler [19], can only be proved to be asymptotically optimal under a certain assumption on the distributions ((12) below). We first show that under the same assumption this test, which we refer to as the *Chernoff* test, is asymptotically optimal in a

strong sense, using the notion of decision making risk in place of the overall probability of error.

- We dispense with the aforementioned assumption by using a modified version of the Chernoff test described in Appendix B2, where we outline the achievability proof of asymptotic optimality without (12).
- We design another test to meet hard risk constraints while retaining asymptotic optimality.

Let \mathcal{F}_k denote the σ -field generated by (Y^k, U^k) . A sequential test $\gamma = (\phi, N, \delta)$ consists of a causal observation control policy ϕ , an \mathcal{F}_k -stopping time N representing the (random) number of observations before the final decision, and the decision rule $\delta = \delta(Y^N, U^N)$. Akin to the paragraph containing (3), the causal control policy ϕ is described by the pmfs $q(u_1), \{q(u_k | y^{k-1}, u^{k-1})\}_{k=2}^\infty$.

A. Chernoff Test

We first present the Chernoff test [18], [19] for sequential design of experiments with multiple hypotheses. The proof of asymptotic optimality of this test requires the following technical assumption which was also imposed in [18], [19]: For every $u \in \mathcal{U}$, $0 \leq i < j < M - 1$,

$$D(p_i^u \| p_j^u) > 0. \quad (12)$$

The Chernoff test admits the following sequential description. Having fixed the control policy up to time k and obtained the first k observations and control values y^k, u^k , if the controller decides to continue taking more observations, then at time $k + 1$, a *randomized* control policy is adopted wherein $U_{k+1} \in \mathcal{U}$ is drawn from the following distribution

$$q(u) = q(u | \hat{i}_k) = \arg \max_{\bar{q}(u)} \min_{j \in \mathcal{M} \setminus \{\hat{i}_k\}} \sum_u \bar{q}(u) D(p_{\hat{i}_k}^u \| p_j^u), \quad (13)$$

where $\hat{i}_k = \arg \max_{i \in \mathcal{M}} p_i(y^k, u^k)$, is the ML estimate of the hypothesis at time k . The stopping rule is defined as the first time n for which

$$\log \left(\frac{p_{\hat{i}_n}(y^n, u^n)}{\max_{j \neq \hat{i}_n} p_j(y^n, u^n)} \right) \geq -\log(c), \quad (14)$$

where c is a positive real-valued parameter that will be selected to approach zero in order to drive the probabilities of error to zero. At the stopping time n , the decision rule is ML, i.e., $\delta(y^n, u^n) = \hat{i}_n$. Note that randomization is used in the causal control policy. This facilitates the simultaneous minimization of the expected stopping time under the M hypotheses as the error probability goes to zero. Also similar to the test proposed in the fixed sample size setting, this sequential test relies on the separation principle between estimation and control, with the distinction that the stationary mapping from the posterior distribution of the hypothesis to the control value is now randomized.

To dispense with (12), we propose a “modified Chernoff test” with a control policy that is slightly different from (13). Specifically, instead of using the policy (13) at all times, we will occasionally sample from the uniform control independently of the index of the ML hypothesis; the specific way in which this is

done will be explained in Appendix B2. The stopping rule of this modify test will still be as in (14) with the same c therein.

B. Asymptotic Optimality

In order to present a formal statement establishing the strong asymptotic optimality of the Chernoff test, we introduce the concept of decision risks or frequentist error probabilities [16]. In particular, let $\pi(i)$, $i \in \mathcal{M}$, be a prior distribution of the hypothesis with a full support. For each $i \in \mathcal{M}$, the *probability of incorrectly deciding i or the risk of deciding i* is given by

$$R_i \triangleq \sum_{j \in \mathcal{M} \setminus \{i\}} \pi(j) \mathbb{P}_j \{\delta = i\}. \quad (15)$$

Note that for each $i \in \mathcal{M}$,

$$R_i = \sum_{j \in \mathcal{M} \setminus \{i\}} \pi(j) \mathbb{P}_j \{\delta = i\} \leq \max_{k \in \mathcal{M}} \mathbb{P}_k \{\delta \neq k\}, \quad (16)$$

Therefore, the condition $\max_{k \in \mathcal{M}} \mathbb{P}_k \{\delta \neq k\} \rightarrow 0$ implies that $\max_{k \in \mathcal{M}} R_k \rightarrow 0$.

Theorem 3: The modified Chernoff test (as $c \rightarrow 0$) satisfies

$$\lim_{c \rightarrow 0} \max_{i \in \mathcal{M}} \mathbb{P}_i \{\delta(Y^N, U^N) \neq i\} = 0, \quad (17)$$

and for each $i \in \mathcal{M}$,

$$\mathbb{E}_i[N] \leq \frac{-\log \left(\max_{k \in \mathcal{M}} \mathbb{P}_k \{\delta \neq k\} \right)}{\max_{q(u)} \min_{j \in \mathcal{M} \setminus \{i\}} \sum_u q(u) D(p_j^u \| p_j^u)} (1 + o(1)) \quad (18)$$

$$\leq \frac{-\log(R_i)}{\max_{q(u)} \min_{j \in \mathcal{M} \setminus \{i\}} \sum_u q(u) D(p_j^u \| p_j^u)} (1 + o(1)). \quad (19)$$

Furthermore, the modified Chernoff test is asymptotically optimal in the following *strong sense*. If the prior π has full support on \mathcal{M} , then any sequence of tests with vanishing maximal risk, i.e., $\max_{k \in \mathcal{M}} R_k \rightarrow 0$, satisfies for *every* $i \in \mathcal{M}$,

$$\mathbb{E}_i[N] \geq \frac{-\log(R_i)}{\max_{q(u)} \min_{j \in \mathcal{M} \setminus \{i\}} \sum_u q(u) D(p_j^u \| p_j^u)} (1 + o(1)). \quad (20)$$

Remark 5: The converse assertion (20) in terms of maximal risk implies the one in terms of the maximal error probability, but not vice versa. Thus the asymptotic optimality of the modified Chernoff test established in Theorem 3 is stronger than the corresponding result in [18], [19], which is given in terms of maximal error probability.

C. Asymptotically Optimal Test Meeting Hard Constraints on the Risks

Although the calculation of risks involves the prior distribution of the hypothesis, the test proposed in Section IV-A does not use the knowledge of the prior distribution at all. In this section, we show that by using this knowledge, we can further modify our test to meet hard constraints on the risks. Another key to this new test is the use of different thresholds for the peak of the posterior distribution depending on the index of the ML hypothesis instead of a single threshold as in (14). In the asymptotic regime

TABLE I
EXAMPLE

	$u = a$	$u = b$	$u = c$
$i = 0$	p	\bar{p}	\bar{p}
$i = 1$	\bar{p}	p	\bar{p}
$i = 2$	\bar{p}	\bar{p}	p

in which all the risks vanish, we show that this modified test will also be asymptotically optimal.

Specifically, for a given tuple $(\bar{R}_1, \dots, \bar{R}_M)$, we will design a test to satisfy $R_i \leq \bar{R}_i$, $i \in \mathcal{M}$. To this end, we modify the stopping rule (14) to be so that we stop at the first time n when

$$\log \left(\frac{\pi(\hat{i}_n) p_{\hat{i}_n}(y^n, u^n)}{\max_{j \neq \hat{i}_n} \pi(j) p_j(y^n, u^n)} \right) \geq \log \left(\frac{(M-1)\pi(\hat{i}_n)}{\bar{R}_{\hat{i}_n}} \right). \quad (21)$$

Theorem 4: For any tuple $(\bar{R}_1, \dots, \bar{R}_M)$, $\bar{R}_i > 0$, $i \in \mathcal{M}$ and any π with a full support, the modified Chernoff test but with the stopping rule (21) in place of (14) satisfies, for every $i \in \mathcal{M}$,

$$\sum_{j \neq i} \pi(j) \mathbb{P}_j \{\delta(Y^N, U^N) = i\} \leq \bar{R}_i. \quad (22)$$

Furthermore, as $\max_{i \in \mathcal{M}} \bar{R}_i \rightarrow 0$, while satisfying $\max_{i \in \mathcal{M}} \bar{R}_i \leq K(\min_{i \in \mathcal{M}} \bar{R}_i)$ for some $K > 0$, the proposed test is asymptotically optimal, i.e., it satisfies (18) and, hence, also (19).

V. EXAMPLE

We consider an example with parameters $M = 3$, $\mathcal{Y} = \{0, 1\}$, $\mathcal{U} = \{a, b, c\}$. For an arbitrary ϵ , $0 < \epsilon < (1/2)$, denote by $p(y)$ and $\bar{p}(y)$, the two pmfs on \mathcal{Y} for which $p(1) = \epsilon$ and $\bar{p}(1) = 1 - \epsilon$, respectively. Then, consider the model for controlled sensing for hypothesis testing in which the pmfs p_i^u , $i \in \{0, 1, 2\}$, $u \in \{a, b, c\}$, are assigned according to Table I.

This example is motivated by adaptive sensor selection for event detection. Consider a sensor network with a fusion center and three sensors a , b and c , collecting measurements from three separate locations 0, 1, 2. A specific event takes place at exactly one unknown location; it affects the distribution of the measurements at this particular location (represented by the distribution p in Table I), while the measurements at the other two locations are distributed according to \bar{p} . At every time step, the fusion center can query only one sensor to measure its readings. The goal is to determine the location of the event in the most efficient manner.

The optimal exponent for open-loop control (cf. (9)) can be easily calculated to be

$$\beta_{\text{OL}} = \frac{2}{3} C(p, \bar{p}) = -\frac{2}{3} \log \left(2\sqrt{\epsilon(1-\epsilon)} \right). \quad (23)$$

For causal control, we apply the control policy presented in Section III-B-2 (cf. (10)). Then, by solving the maximization in (10), we obtain a *deterministic* causal control policy, which is given by $u_{k+1} = f(\hat{i}_k)$, where $f(0) = a$, $f(1) = b$, $f(2) = c$. Lastly, at time n , the decision is made for the maximum likelihood estimate, i.e., $\delta(y^n) = \hat{i}_n$. We now analyze the maximal

error probability of this test. To this end, for any y^n , we let

$$k_a = \left| \left\{ k \in \{1, \dots, n\} : \begin{array}{l} u_k = a \text{ and } y_k = 1, \text{ or} \\ u_k \neq a \text{ and } y_k = 0 \end{array} \right\} \right|. \quad (24)$$

Then, we get from Table I that $p_0(y^n) = \epsilon^{k_a}(1 - \epsilon)^{n-k_a}$. Similarly, we can define k_b and k_c with a in (24) replaced by b and c , respectively, and get that $p_1(y^n) = \epsilon^{k_b}(1 - \epsilon)^{n-k_b}$, $p_2(y^n) = \epsilon^{k_c}(1 - \epsilon)^{n-k_c}$.

We sort $\{k_a, k_b, k_c\}$ in an ascending order and denote the sorted values by $k_1 \leq k_2 \leq k_3$. Note that at every time step, the most likely hypothesis is the one associated with k_1 . Then, it follows from Table I that as n increases by one, if $y_n = 1$, then *the least* of $\{k_a, k_b, k_c\}$ increases by one, while the other two remain fixed. On the other hand, if $y_n = 0$, then *the least* of $\{k_a, k_b, k_c\}$ remains fixed, while the other two increase by one. Hence, If we let k denote the number of zeros in y^n , then $(k_a + k_b + k_c)/3 = (n + k)/3$. In addition, starting from no observation at time zero when $\{k_a, k_b, k_c\}$ are all equal to zero, we get from an induction argument that, $k_2 \leq k_3 \leq k_2 + 1$. This argument is similar to that in [pp. 54] [9]; we refer the reader to [9] for further details. We can now conclude from these previous identities that

$$k_2 \geq \frac{k_a + k_b + k_c}{3} - \frac{1}{3} = \frac{n + k}{3} - \frac{1}{3}. \quad (25)$$

At time n , $\delta(y^n)$ corresponds to the smallest k_1 ; it follows from (25) that for any $i = 0, 1, 2$,

$$\begin{aligned} \mathbb{P}_i\{\delta \neq i\} &\leq \sum_{y^n} \epsilon^{k_2(y^n)} (1 - \epsilon)^{n-k_2(y^n)} \\ &= \sum_{w=1}^n \sum_{y^n: |\{k: y_k=0\}|=w} \epsilon^{k_2(y^n)} (1 - \epsilon)^{n-k_2(y^n)} \\ &\leq \left(\sum_{w=0}^n \binom{n}{w} \epsilon^{\frac{(n+w)}{3} - \frac{1}{3}} (1 - \epsilon)^{\frac{(2n-w)}{3} + \frac{1}{3}} \right) \\ &= \frac{\left(\epsilon^{\frac{1}{3}} (1 - \epsilon)^{\frac{2}{3}} + \epsilon^{\frac{2}{3}} (1 - \epsilon)^{\frac{1}{3}} \right)^n}{\epsilon^{\frac{1}{3}} (1 - \epsilon)^{-\frac{1}{3}}}, \end{aligned}$$

and we get that

$$\begin{aligned} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \left(\max_{i=0,1,2} \mathbb{P}_i\{\delta \neq i\} \right) \\ \geq -\log \left(\epsilon^{\frac{1}{3}} (1 - \epsilon)^{\frac{2}{3}} + \epsilon^{\frac{2}{3}} (1 - \epsilon)^{\frac{1}{3}} \right). \quad (26) \end{aligned}$$

Comparing (23) to (26), we get that for every ϵ , $0 < \epsilon < (1/2)$, causal control can yield a larger error exponent than the best open-loop control. By the symmetry in Table I, the upper bound for β_C in (11) can be calculated to be $C(p, \overline{p}) = -\log(2\sqrt{\epsilon(1-\epsilon)})$.

Lastly, we show that our lower bound for β_C in (11) gives the same achievable error exponent in (26) for this example. To this end, we consider the argument of the $-\log$ in the lower bound

$$\sup_{\nu} \min_u \max_i \left(\sum_y p_i^u(y) e^{\frac{\eta(\nu \circ p^u(y) - p_i^u(y))}{(1-\nu(i))}} \right). \quad (27)$$

Note that the argument minimizer u in (27) is a function of ν . Hence, if the minimizer is replaced by a specific function $u = f(\nu)$, then we will get a larger quantity, i.e.,

$$\begin{aligned} \sup_{\nu} \min_u \max_i \left(\sum_y p_i^u(y) e^{\frac{\eta(\nu \circ p^u(y) - p_i^u(y))}{(1-\nu(i))}} \right) \\ \leq \sup_{\nu} \max_i \left(\sum_y p_i^{f(\nu)}(y) e^{\frac{\eta(\nu \circ p^{f(\nu)}(y) - p_i^{f(\nu)}(y))}{(1-\nu(i))}} \right). \quad (28) \end{aligned}$$

In particular, consider the following function

$$u = f(\nu) = \begin{cases} a, & \arg \max_i \nu(i) = 0, \\ b, & \arg \max_i \nu(i) = 1, \\ c, & \arg \max_i \nu(i) = 2. \end{cases} \quad (29)$$

For an arbitrary $\nu(i)$, $i = 0, 1, 2$, denote their respective sorted values by $\nu_u \geq \nu_c \geq \nu_\ell$. Then, it follows from (29) via appropriate algebraic manipulations using Table I that

$$\begin{aligned} \sup_{\nu} \max_i \left(\sum_y p_i^{f(\nu)}(y) e^{\frac{\eta(\nu \circ p^{f(\nu)}(y) - p_i^{f(\nu)}(y))}{(1-\nu(i))}} \right) \\ = \sup_{1 > \nu_u \geq \nu_c \geq \nu_\ell} \max \left(\begin{array}{l} (1 - \epsilon)e^{-\eta(1-2\epsilon)} + \epsilon e^{\eta(1-2\epsilon)}, \\ (1 - \epsilon)e^{\frac{-\eta(1-2\epsilon)\nu_u}{(1-\nu_c)}} + \epsilon e^{\frac{\eta(1-2\epsilon)\nu_u}{(1-\nu_c)}}, \\ (1 - \epsilon)e^{\frac{-\eta(1-2\epsilon)\nu_u}{(1-\nu_\ell)}} + \epsilon e^{\frac{\eta(1-2\epsilon)\nu_u}{(1-\nu_\ell)}} \end{array} \right). \quad (30) \end{aligned}$$

Next, we select $\eta = 2 \log((1 - \epsilon)/\epsilon)/3(1 - 2\epsilon)$. Note that for any $\nu_u \geq \nu_c \geq \nu_\ell$,

$$\frac{1}{3} \leq \frac{2\nu_u}{3(1 - \nu_\ell)} \leq \frac{2\nu_u}{3(1 - \nu_c)} \leq \frac{2}{3}. \quad (31)$$

It then follows from the selection of η , (31), and the fact that for any $0 < \epsilon < (1/2)$,

$$\max_{\frac{1}{3} \leq s \leq \frac{2}{3}} (1 - \epsilon)^{1-s} \epsilon^s + (1 - \epsilon)^s \epsilon^{1-s} = (1 - \epsilon)^{\frac{2}{3}} \epsilon^{\frac{1}{3}} + (1 - \epsilon)^{\frac{1}{3}} \epsilon^{\frac{2}{3}},$$

that for any $\nu_u \geq \nu_c \geq \nu_\ell$,

$$\max \left(\begin{array}{l} (1 - \epsilon)e^{-\gamma} + \epsilon e^{\gamma}, \\ (1 - \epsilon)e^{-\gamma_c} + \epsilon e^{\gamma_c}, \\ (1 - \epsilon)e^{-\gamma_\ell} + \epsilon e^{\gamma_\ell} \end{array} \right) = (1 - \epsilon)^{\frac{2}{3}} \epsilon^{\frac{1}{3}} + (1 - \epsilon)^{\frac{1}{3}} \epsilon^{\frac{2}{3}},$$

where $\gamma = 2 \log((1 - \epsilon)/\epsilon)/3$, $\gamma_c = 2 \log((1 - \epsilon)/\epsilon)\nu_u/3(1 - \nu_c)$, $\gamma_\ell = 2 \log((1 - \epsilon)/\epsilon)\nu_u/3(1 - \nu_\ell)$. Following from (28) and (30) by taking $-\log$, we get that

$$\begin{aligned} \beta_C &\geq -\log \left(\sup_{\nu} \max_i \left(\sum_y p_i^{f(\nu)}(y) e^{\frac{\eta(\nu \circ p^{f(\nu)}(y) - p_i^{f(\nu)}(y))}{(1-\nu(i))}} \right) \right) \\ &= -\log \left((1 - \epsilon)^{\frac{2}{3}} \epsilon^{\frac{1}{3}} + (1 - \epsilon)^{\frac{1}{3}} \epsilon^{\frac{2}{3}} \right) \end{aligned}$$

as required. This lower bound matches the one in (26).

In the sequential setting, the quantities dictating the asymptotically optimal performance are $\max_{q(u)} \min_{j \in \mathcal{M} \setminus \{i\}} \sum_u q(u) D(p_i^u \| p_j^u)$, the denominators on the right-side of (18), which can readily be computed for this example to be $-\log(2\sqrt{\epsilon(1-\epsilon)})$ for every $i \in \mathcal{M}$. The numerical value of this quantity is, as expected, larger than

β_C in the fixed sample size setting, as now the control has an additional capability to adaptively stop taking observations based on past observations.

VI. DISCUSSION

In the proposed sequential test in Section IV, information from the past is used to form the maximum likelihood estimate of the hypothesis, which is used in turn to select the maximizing distribution and the maximizing control value in (13). In contrast to binary hypothesis testing with a fixed sample size (cf. Proposition 1), information from the past seems to be crucial for attaining the asymptotically optimal performance in the sequential setting, since the mentioned maximizers can depend on the identity of the ML hypothesis even for the case of binary hypothesis testing.

VII. CONCLUSIONS

We studied the structure of the optimal controller for multihypothesis testing with observation control under various asymptotic regimes. First, in a setting with a fixed sample size, the optimal error exponent corresponds to the maximum Chernoff information over the choice of controls for binary hypothesis testing. In particular, in this setup, a pure stationary open-loop control policy is asymptotically optimal even among the broader class of causal control policies. For multiple hypothesis testing, we characterized the optimal error exponent achievable by open-loop control and derived precise lower and upper bounds for the optimal error exponent achievable by causal control. We also proposed a causal control policy for multihypothesis testing based on maximizing the minimum Chernoff information of the distributions corresponding to the most likely hypothesis and all the alternative hypotheses. We illustrated through an example that the proposed causal control policy strictly outperforms the best open-loop control policy.

Second, we considered a sequential setting wherein the objective is to minimize the expected stopping time subject to the constraints of vanishing error probabilities under each hypothesis. We proposed a suitably modified version of the Chernoff test for multiple hypotheses testing and showed that it is asymptotically optimal in a strong sense, using the notion of decision making risk instead of the overall probability of error. Our control policy is based on maximizing the KL distance of the distributions corresponding to the most likely hypothesis and the nearest alternative hypothesis. We also designed another sequential test to meet hard constraints on the risks while retaining the asymptotic optimality.

For binary hypothesis testing, the findings showed that past information is crucial in achieving the asymptotically optimal performance in the sequential setting, while it is superfluous in the fixed sample size setting. Our results also showed that for general multiple hypothesis testing, randomization in control is always superfluous (for any number of hypotheses) in achieving the asymptotically optimal performance in the fixed sample size setting. On the other hand, we showed that in the sequential setting, randomization can facilitate the structure of the asymptotically optimal control policy following the separation principle between estimation and control especially in the sequential setting.

APPENDIX

A. Proof of Results in Section III

1) *Proof of Theorem 1:* We start with the achievability proof. First, note that for any n , and any test

$$\begin{aligned} \frac{1}{M} \sum_{i \in \mathcal{M}} \mathbb{P}_i \{\delta \neq i\} &\leq \max_{i \in \mathcal{M}} \mathbb{P}_i \{\delta \neq i\} \\ &\leq M \left(\frac{1}{M} \sum_{i \in \mathcal{M}} \mathbb{P}_i \{\delta \neq i\} \right). \end{aligned} \quad (32)$$

Fix a sequence $u^n \in \mathcal{U}^n$, and let $\delta_{\text{ML}} : \mathcal{Y}^n \rightarrow \mathcal{M}$ be the ML decision rule. It now follows that

$$\begin{aligned} \frac{1}{M} \sum_{i \in \mathcal{M}} \mathbb{P}_i \{\delta_{\text{ML}}(Y^n) \neq i\} \\ = \frac{1}{M} \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{M} \setminus \{i\}} \mathbb{P}_i \{\delta_{\text{ML}}(Y^n) = j\}. \end{aligned} \quad (33)$$

For any $i, j, 0 \leq i < j \leq M-1$, and any $s \in [0, 1]$, we get that

$$\mathbb{P}_i \{\delta_{\text{ML}}(Y^n) = j\} \leq \mathbb{P}_i \left\{ p_i(y^n)^s p_j(y^n)^{1-s} \geq p_i(y^n) \right\}, \quad (34)$$

and

$$\mathbb{P}_j \{\delta_{\text{ML}}(Y^n) = i\} \leq \mathbb{P}_j \left\{ p_i(y^n)^s p_j(y^n)^{1-s} \geq p_j(y^n) \right\}. \quad (35)$$

Combining (34) and (35), we obtain that

$$\begin{aligned} \mathbb{P}_i \{\delta_{\text{ML}}(Y^n) = j\} + \mathbb{P}_j \{\delta_{\text{ML}}(Y^n) = i\} \\ \leq \int_{y^n} \prod_{k=1}^n (p_i^{u_k}(y_k)^s p_j^{u_k}(y_k)^{1-s}) \prod_{k=1}^n d\mu_{u_k}(y_k) \\ = \prod_{k=1}^n \left(\int_{y_k} p_i^{u_k}(y_k)^s p_j^{u_k}(y_k)^{1-s} d\mu_{u_k}(y_k) \right) \\ = e^{n \left(\sum_{u \in \mathcal{U}} \bar{q}(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right) \right)}, \end{aligned} \quad (36)$$

where $\bar{q}(\cdot)$ denotes the empirical distribution of u^n : $\bar{q}(u) \triangleq (1/n) |\{k : k \in \{1, \dots, n\}, u_k = u\}|$. Since (36) is true for any $s \in [0, 1]$, we get that

$$\begin{aligned} \mathbb{P}_i \{\delta_{\text{ML}}(Y^n) = j\} + \mathbb{P}_j \{\delta_{\text{ML}}(Y^n) = i\} \\ \leq e^{-n \left(\max_{s \in [0, 1]} \left(\sum_{u \in \mathcal{U}} \bar{q}(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right) \right) \right)}. \end{aligned}$$

Because there are only finitely many pairs of hypotheses in the sum on the right-side of (33), the pair corresponding to the smallest exponent will dominate the exponent. Hence, we get

$$\begin{aligned} \frac{1}{M} \sum_{i \in \mathcal{M}} \mathbb{P}_i \{\delta_{\text{ML}}(Y^n) \neq i\} &\leq (M-1) \\ &\times e^{-n \left(\min_{i < j} \max_{s \in [0, 1]} \left(\sum_{u \in \mathcal{U}} \bar{q}(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right) \right) \right)}. \end{aligned}$$

Since u^n is arbitrary, we can approximate any distribution $q(u)$ arbitrarily close by the empirical distribution

$\bar{q}^{(n)}(u)$ of an appropriate *deterministic* sequence u^n such that $\max_u |\bar{q}^{(n)}(u) - q(u)| \rightarrow 0$. This fact combined with (32) yields that

$$\beta_{\text{OL}} \geq \max_{q(u)} \min_{i < j} \max_{s \in [0,1]} - \sum_u q(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right), \quad (37)$$

and that the error exponent on the right-side of (37) is achievable by pure open-loop control.

Next, we prove that the reverse inequality of (37). Since we proved that β_{OL} is achievable by pure control, we restrict our attention to pure open-loop control. By considering the necessary and sufficient condition for the maximizing s of the function

$$- \sum_{k=1}^n \log \left(\int_{y_k} p_i^{u_k}(y_k)^s p_j^{u_k}(y_k)^{1-s} d\mu_{u_k}(y_k) \right),$$

we obtain for any $u^n \in \mathcal{U}^n$, and any pair of hypotheses $i, j \in \mathcal{M}$ that

$$s^* = \arg \max_{s \in [0,1]} - \sum_{k=1}^n \log \left(\int_{y_k} p_i^{u_k}(y_k)^s p_j^{u_k}(y_k)^{1-s} d\mu_{u_k}(y_k) \right) \quad (38)$$

satisfies (cf. (5))

$$\begin{aligned} \max_{s \in [0,1]} - \sum_{k=1}^n \log \left(\int_{y_k} p_i^{u_k}(y_k)^s p_j^{u_k}(y_k)^{1-s} d\mu_{u_k}(y_k) \right) \\ = \sum_{k=1}^n D(b_{ij}^{u_k, s^*} \| p_i^{u_k}) = \sum_{k=1}^n D(b_{ij}^{u_k, s^*} \| p_j^{u_k}). \end{aligned} \quad (39)$$

We next consider, for the same pair of hypotheses i, j as above, the pdf/pmf \tilde{p} defined by $\tilde{p}(y^n) \triangleq \prod_{k=1}^n b_{ij}^{u_k, s^*}(y_k)$. For any test, it either holds that

$$\tilde{\mathbb{P}}\{\delta(Y^n) = i\} \geq \frac{1}{2}, \text{ or that } \tilde{\mathbb{P}}\{\delta(Y^n) \neq i\} \geq \frac{1}{2}. \quad (40)$$

Suppose that the first case of (40) holds. For any causal control policy, under the stationary Markovity assumption and assumption (2), it follows that the random process S_k , $k = 1, \dots, n$, where

$$S_k \triangleq \sum_{l=1}^k \left(\log \left(\frac{b_{ij}^{u_l, s^*}(Y_l)}{p_j^{u_l}(Y_l)} \right) - \mathbb{E} \left[\log \left(\frac{b_{ij}^{u_l, s^*}(Y_l)}{p_j^{u_l}(Y_l)} \right) \middle| \mathcal{F}_{l-1} \right] \right),$$

is a “stable” martingale adapted to \mathcal{F}_k , the sigma fields generated by (Y^k, U^k) , $k = 1, \dots, n$. By the martingale stability theorem of Loève [26, pp. 53], we get that $\{(1/n)S_n\}_{n=1}^\infty$ converges to zero a.s. and, hence, in probability, i.e., for any $\eta > 0$,

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}} \left\{ \frac{1}{n} \sum_{k=1}^n \left(\log \left(\frac{b_{ij}^{u_k, s^*}(Y_k)}{p_j^{u_k}(Y_k)} \right) - \mathbb{E} \left[\log \left(\frac{b_{ij}^{u_k, s^*}(Y_k)}{p_j^{u_k}(Y_k)} \right) \middle| \mathcal{F}_{k-1} \right] \right) > \eta \right\} = 0. \quad (41)$$

Since u_k , $k = 1, \dots, n$ are fixed (pure open-loop control policy), we obtain from (41) that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}} \left\{ \frac{1}{n} \sum_{k=1}^n \left(\log \left(\frac{b_{ij}^{u_k, s^*}(Y_k)}{p_j^{u_k}(Y_k)} \right) - D(b_{ij}^{u_k, s^*} \| p_j^{u_k}) \right) > \eta \right\} = 0. \quad (42)$$

The first inequality of (40) and (42) yield that for any $\epsilon' > 0$, any $\eta > 0$ and all n large,

$$\begin{aligned} \frac{1}{2} - \epsilon' \leq \tilde{\mathbb{P}} \left\{ \delta(Y^n) = i, \prod_{k=1}^n p_j^{u_k}(Y_k) \right. \\ \left. > e^{-n \left(\sum_{k=1}^n \frac{1}{n} D(b_{ij}^{u_k, s^*} \| p_j^{u_k}) \right) + \eta} \right. \\ \left. \times \prod_{k=1}^n b_{ij}^{u_k, s^*}(Y_k) \right\} \\ \leq \mathbb{P}_j \{ \delta(Y^n) \neq j \} e^{n \left(\sum_{k=1}^n \frac{1}{n} D(b_{ij}^{u_k, s^*} \| p_j^{u_k}) + \eta \right)}. \end{aligned} \quad (43)$$

If the second case of (40) holds instead, then similar to (41) we obtain that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{P}} \left\{ \frac{1}{n} \sum_{k=1}^n \left(\log \left(\frac{b_{ij}^{u_k, s^*}(Y_k)}{p_i^{u_k}(Y_k)} \right) - D(b_{ij}^{u_k, s^*} \| p_i^{u_k}) \right) > \eta \right\} = 0. \quad (44)$$

From the second case of (40) and (44), we obtain that for any $\epsilon' > 0$ and any $\eta > 0$,

$$\frac{1}{2} - \epsilon' \leq \mathbb{P}_i \{ \delta(Y^n) \neq i \} e^{n \left(\sum_{k=1}^n \frac{1}{n} D(b_{ij}^{u_k, s^*} \| p_i^{u_k}) + \eta \right)}, \quad (45)$$

which parallels (43). It now follows from (43), (45) and (39) that for any $i, j \in \mathcal{M}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \left(\max_{i \in \mathcal{M}} \mathbb{P}_i \{ \delta \neq i \} \right) \\ \leq \lim_{n \rightarrow \infty} -\frac{1}{n} \log (\max (\mathbb{P}_i \{ \delta(Y^n) \neq i \}, \mathbb{P}_j \{ \delta(Y^n) \neq j \})) \\ \leq \max \left(\frac{1}{n} \sum_{k=1}^n D(b_{ij}^{u_k, s^*} \| p_i^{u_k}), \frac{1}{n} \sum_{k=1}^n D(b_{ij}^{u_k, s^*} \| p_j^{u_k}) \right) \\ = \frac{1}{n} \sum_{k=1}^n -\log \left(\int_{y_k} p_i^{u_k}(y_k)^{s^*} p_j^{u_k}(y_k)^{1-s^*} d\mu_{u_k}(y_k) \right) \\ = -\sum_u \bar{q}(u) \log \left(\int_y p_i^u(y)^{s^*} p_j^u(y)^{1-s^*} \mu_u(y) \right) \\ = \max_{s \in [0,1]} -\sum_u \bar{q}(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} \mu_u(y) \right), \end{aligned} \quad (46)$$

where \bar{q} denotes the empirical distribution of u^n . Since (46) must hold for every pair i, j of hypotheses, we then get that

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \left(\max_{i \in \mathcal{M}} \mathbb{P}_i \{ \delta(Y^n) \neq i \} \right) \leq \min_{i < j} \max_{s \in [0,1]} - \sum_u \bar{q}(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right),$$

and, hence,

$$\beta_{OL} \leq \max_{q(u)} \min_{i < j} \max_s - \sum_u q(u) \log \left(\int_y p_i^u(y)^s p_j^u(y)^{1-s} d\mu_u(y) \right). \quad (47)$$

Note that in (46), the empirical distribution $\bar{q}(u)$ depends only on the pure control u^n and *not* on the pair of hypotheses i, j , while the maximizer s^* in (38) depends *both* on u^n and on the pair of hypotheses. The assertion of Theorem 2 is now proved by combining (37) and (47).

2) *Proof of Theorem 2:* We first prove that β_C is achievable by a pure control policy. For any fixed n , the problem of finding the optimal causal control that minimizes the *exact* average probability of error can be cast as a finite-horizon stochastic optimal control problem through the use of the posterior distribution as a sufficient statistic. Since \mathcal{U} is finite, it follows from a standard dynamic programming argument [27] that the optimal causal control is a deterministic one.

Next, we prove the upper bound for β_C in (11). Observe that for any test for M hypotheses, with a decision rule δ and any pair of hypotheses $i, j \in \mathcal{M}$, a binary test for hypotheses i and j , can be constructed using the same control policy and an appropriate decision rule $\tilde{\delta}$ so that

$$\max \left(\mathbb{P}_i \{ \tilde{\delta}(Y^n, U^n) \neq i \}, \mathbb{P}_j \{ \tilde{\delta}(Y^n, U^n) \neq j \} \right) \leq \max_{i \in \mathcal{M}} \mathbb{P}_i \{ \delta(Y^n, U^n) \neq i \}.$$

Applying the converse part of Theorem 1 with the roles of $\{p_0^u\}_{u \in \mathcal{U}}$ and $\{p_1^u\}_{u \in \mathcal{U}}$ therein being played by $\{p_i^u\}_{u \in \mathcal{U}}$ and $\{p_j^u\}_{u \in \mathcal{U}}$, respectively, we obtain that

$$\beta_C \leq \max_{u \in \mathcal{U}} \max_{s \in [0,1]} - \log \left(\int_y p_i^u(x)^s p_j^u(y)^{1-s} d\mu_u(y) \right).$$

As the previous argument applies for any $i \neq j, i, j \in \mathcal{M}$, we obtain the upper bound in (11) by minimizing over all pairs of hypotheses $i, j \in \mathcal{M}$.

It is then only left to prove the lower bound for β_C in (11). The proof relies on the following lemma whose proof is deferred to Appendix A3.

Lemma 1: Let $J = |\mathcal{Y}|$. For every $\epsilon, 0 < \epsilon < 1$, and $\eta > 0$, it holds that

$$\sup_{\nu} \min_u \max_i \sum_y p_i^u(y) \left(\frac{1 + \epsilon (J p_i^u(y) - 1)}{1 + \epsilon \left(\sum_{j \neq i} \frac{J \nu(j) p_j^u(y)}{(1 - \nu(i))} - 1 \right)} \right)^{-\frac{\eta}{J\epsilon}} \geq e^{-\beta_C}, \quad (48)$$

where the outer supremum on the left-side of (48) is over the set of all pmfs on \mathcal{M} that are not point-mass distributions.

By L'Hôpital's rule, for every ν that is not a point-mass distribution,

$$\lim_{\epsilon \rightarrow 0} \left(\frac{1 + \epsilon (J p_i^u(y) - 1)}{1 + \epsilon \left(\sum_{j \neq i} \frac{J \nu(j) p_j^u(y)}{(1 - \nu(i))} - 1 \right)} \right)^{-\frac{\eta}{J\epsilon}} \rightarrow e^{\frac{\eta \left(\sum_{j \neq i} \frac{\nu(j) p_j^u(y)}{(1 - \nu(i))} - p_i^u(y) \right)}{(1 - \nu(i))}} = e^{\frac{\eta (\nu \circ p^u(y) - p_i^u(y))}{(1 - \nu(i))}}. \quad (49)$$

Consequently, by letting $\epsilon \rightarrow 0$, we get from Lemma 1 and (49) through the finiteness of $\mathcal{M}, \mathcal{U}, \mathcal{Y}$ that for any $\eta > 0$,

$$- \log \left(\sup_{\nu} \min_u \max_i \left(\sum_y p_i^u(y) e^{\frac{\eta (\nu \circ p^u(y) - p_i^u(y))}{(1 - \nu(i))}} \right) \right) \leq \beta_C.$$

The proof of Theorem 3 follows by optimizing over $\eta > 0$.

3) *Proof of Lemma 1:* We shall consider a test based on a mismatched posterior distribution on the hypothesis. In particular, the control value at every time is picked based on the posterior distribution on \mathcal{M} computed based on an appropriately chosen mismatched model $\{q_i^u\}_{i \in \mathcal{M}}^{u \in \mathcal{U}}$ (instead of the real model $\{p_i^u\}_{i \in \mathcal{M}}^{u \in \mathcal{U}}$) and the uniform prior distribution on \mathcal{M} . In particular, denote the posterior probability of hypothesis $i \in \mathcal{M}$ at time $k = 0, \dots, n$, by $\nu_k(i)$. Then,

$$\nu_0(i) = \frac{1}{M}, \quad \nu_k(i) = \frac{\prod_{l=1}^k q_i^{u_l(y^{l-1})}(y_l)}{\sum_j \prod_{l=1}^k q_j^{u_l(y^{l-1})}(y_l)}, \quad 1 \leq k \leq n.$$

Also denote the likelihood ratio for hypothesis $i \in \mathcal{M}$ at time $k = 0, \dots, n$, by $l_k(i)$, i.e.,

$$l_0(i) = \frac{1}{M-1}, \quad l_k(i) \triangleq \frac{\nu_k(i)}{1 - \nu_k(i)} = \frac{\nu_k(i)}{\sum_{j \neq i} \nu_k(j)},$$

$$l_{k+1}(i) = \frac{\nu_k(i) q_i^{u_{k+1}(\nu_k(y^k))}(y_{k+1})}{\sum_{j \neq i} \nu_k(j) q_j^{u_{k+1}(\nu_k(y^k))}(y_{k+1})}, \quad 0 \leq k \leq n-1. \quad (50)$$

The decision rule at time n is the maximum likelihood estimate of the hypothesis, i.e., $\delta(y^n) = \arg \max_i \nu_n(i)$. Next, we analyze the probability of error of such test as a function of $\{q_i^u\}$, $i \in \mathcal{M}$, $u \in \mathcal{U}$, and the pure control $u_k = u_k(\nu_{k-1}) = u_k(y^{k-1})$ which will be specified later. We get that for any $\lambda < 0$, the probability of error (with respect to the *real* model $\{p_i^u\}$, $i \in \mathcal{M}$, $u \in \mathcal{U}$) under hypothesis i can be upper bounded as

$$\begin{aligned} \mathbb{P}_i\{\delta \neq i\} &= \mathbb{P}_i\left\{\arg \max_j \Pi_n(j) \neq i\right\} \\ &\leq \mathbb{P}_i\{L_n(i) \leq 1\} \leq \mathbb{E}_i[L_n(i)^\lambda]. \end{aligned} \quad (51)$$

Next, by writing

$$L_n(i) = \prod_{k=1}^n \left(\frac{L_k(i)}{L_{k-1}(i)} \right), \quad L_0(i) = \prod_{k=1}^n \left(\frac{L_k(i)}{L_{k-1}(i)} \right) \frac{1}{M-1}, \quad (52)$$

and substituting (52) into (51), we get that for any $\lambda < 0$,

$$\mathbb{P}_i\{\delta \neq i\} \leq \mathbb{E}_i\left[\prod_{k=1}^n \left(\frac{L_k(i)}{L_{k-1}(i)} \right)^\lambda\right] (M-1)^{-\lambda}. \quad (53)$$

We next specify the mismatched model $\{q_i^u\}$, $i \in \mathcal{M}$, $u \in \mathcal{U}$. For any ϵ , $0 < \epsilon < 1$, consider the conditional pmf $W_\epsilon(y|y')$, $y, y' \in \mathcal{Y}$, such that

$$W_\epsilon(y|y') = \begin{cases} \frac{1}{J} + \frac{J-1}{J}\epsilon, & y = y', \\ \frac{1}{J} - \frac{1}{J}\epsilon, & y \neq y'. \end{cases} \quad (54)$$

Then, let

$$\begin{aligned} q_i^u(y) &= p_i^u \circ W_\epsilon(y) \triangleq \sum_{y'} p_i^u(y') W_\epsilon(y|y') \\ &= p_i^u(y) \left(\frac{1}{J} + \frac{(J-1)\epsilon}{J} \right) + \sum_{y' \neq y} p_i^u(y') \left(\frac{1}{J} - \frac{\epsilon}{J} \right) \\ &= \frac{1}{J} + \frac{\epsilon}{J} (J p_i^u(y) - 1). \end{aligned} \quad (55)$$

Using this particular $\{q_i^u\}$, $i \in \mathcal{M}$, $u \in \mathcal{U}$, with \mathcal{F}_{k-1} denoting the sigma field generated by y^{k-1} , $k = 1, \dots, n$, we get from (50) through an easy algebraic manipulation that

$$\begin{aligned} &\mathbb{E}_i\left[\left(\frac{L_k(i)}{L_{k-1}(i)}\right)^\lambda \middle| \mathcal{F}_{k-1}\right] \\ &= \sum_y p_i^{u_k}(y) \left(\frac{(1 - \nu_{k-1}(i)) q_i^{u_k}(y)}{\sum_{j \neq i} \nu_{k-1}(j) q_j^{u_k}(y)} \right)^\lambda \\ &= \sum_y p_i^{u_k}(y) \left(\frac{1 + \epsilon [J p_i^{u_k}(y) - 1]}{1 + \epsilon \left(\sum_{j \neq i} \frac{J \nu_{k-1}(j) p_j^{u_k}(y)}{(1 - \nu_{k-1}(i))} - 1 \right)} \right)^\lambda, \end{aligned}$$

where $u_k = u_k(\nu_{k-1}) = u_k(y^{k-1})$. Next, let $\lambda = -(\eta/J\epsilon)$ for an arbitrary $\eta > 0$, and let

$$\begin{aligned} &u^*(\nu) \\ &= \arg \min_u \max_i \sum_y p_i^u(y) \left(\frac{1 + \epsilon (J p_i^u(y) - 1)}{1 + \epsilon \left(\sum_{j \neq i} \frac{J \nu(j) p_j^u(y)}{(1 - \nu(i))} - 1 \right)} \right)^{-\frac{\eta}{J\epsilon}}. \end{aligned} \quad (56)$$

If we select the control to be $u_k = u^*(\nu_{k-1})$, where $u^*(\nu)$ is as in (56), then we get that for any $i \in \mathcal{M}$, $k = 2, \dots, n$, and any realization of ν_{k-1} (as a function of y^{k-1}),

$$\begin{aligned} &\mathbb{E}_i\left[\left(\frac{L_k(i)}{L_{k-1}(i)}\right)^{-\frac{\eta}{J\epsilon}} \middle| \mathcal{F}_{k-1}\right] \\ &= \min_u \max_i \sum_y p_i^u(y) \left(\frac{1 + \epsilon (J p_i^u(y) - 1)}{1 + \epsilon \left(\sum_{j \neq i} \frac{J \nu_{k-1}(j) p_j^u(y)}{(1 - \nu_{k-1}(i))} - 1 \right)} \right)^{-\frac{\eta}{J\epsilon}}. \end{aligned}$$

Note that since ν_0 (uniform) and all q_i^u , $i \in \mathcal{M}$, $u \in \mathcal{U}$, have full supports (cf. (55) upon noting that $\epsilon < 1$), it follows that for every $k = 1, \dots, n$, and every realization y^k , $\nu_k(y^k)$ will have a full support. With this observation, continuing from (53) by using the smoothing property of conditional expectation, we get that

$$\begin{aligned} e^{-\beta_C} &\leq \left(\max_{i \in \mathcal{M}} \mathbb{P}_i\{\delta \neq i\} \right)^{\frac{1}{n}} \\ &\leq \sup_{\nu} \min_u \max_i \sum_y p_i^u(y) \left(\frac{1 + \epsilon (J p_i^u(y) - 1)}{1 + \epsilon \left(\sum_{j \neq i} \frac{J \nu(j) p_j^u(y)}{(1 - \nu(i))} - 1 \right)} \right)^{-\frac{\eta}{J\epsilon}} \\ &\quad \times (M-1)^{\frac{\eta}{nJ\epsilon}}. \end{aligned}$$

The lemma follows by taking the limit as $n \rightarrow \infty$.

B. Proofs of Results in Section IV

1) *The Converse Proof of Theorem 3:* We now prove the assertion (20). To simplify notation let

$$\begin{aligned} d_i^* &\triangleq \max_{q(u)} \min_{j \in \mathcal{M} \setminus \{i\}} \sum_u q(u) D(p_i^u \| p_j^u), \\ Z_{ij}(n) &\triangleq \log \frac{p_i(Y^n, U^n)}{p_j(Y^n, U^n)} \end{aligned}$$

It is not hard to see that (20) follows immediately from Lemma 2 below and Markov inequality.

Lemma 2: For every $0 < \rho < 1$, any sequence of tests with vanishing maximal risk i.e., $\max_{k \in \mathcal{M}} R_k \rightarrow 0$, satisfies

$$\mathbb{P}_i \left\{ N > \frac{-\rho \log R_i}{d_i^*} \right\} \rightarrow 1,$$

for every $i \in \mathcal{M}$.

Lemma 2 in turn relies on the following lemma.

Lemma 3: For any sequence of tests with $\max_{k \in \mathcal{M}} R_k \rightarrow 0$, any $0 < \rho < 1$, it holds that for each $j \in \mathcal{M}$,

$$\mathbb{P}_i \{Z_{ij}(N) \geq -\rho \log R_i\} \rightarrow 1. \quad (57)$$

Proof: Define the subset Q_n of the sample space as

$$Q_n = \{(y^n, u^n) : Z_{ij}(n) < \rho \log R_i, \delta = i, N = n\}$$

From the definition of R_i in (15), for every $j \in \mathcal{M} \setminus \{i\}$, we have the following set of inequalities

$$\frac{R_i}{\pi(j)} \geq \mathbb{P}_j\{\delta = i\} \geq \sum_{n=1}^{\infty} \mathbb{P}_j\{Q_n\} \geq R_i^{\rho} \sum_{n=1}^{\infty} \mathbb{P}_i\{Q_n\},$$

where the third inequality follows from the fact that $Z_{ij}(n) < -\rho \log R_i$ on Q_n . Hence, for every $i \neq j, i, j \in \mathcal{M}$,

$$\sum_{n=1}^{\infty} \mathbb{P}_i\{Q_n\} \leq \frac{R_i^{1-\rho}}{\pi(j)}. \quad (58)$$

Thus,

$$\begin{aligned} \mathbb{P}_i \{Z_{ij}(N) < -\rho \log R_i\} &\leq \sum_{n=1}^{\infty} \mathbb{P}_i\{Q_n\} + \mathbb{P}_i\{\delta \neq i\} \\ &\leq \frac{R_i^{1-\rho}}{\pi(j)} + \sum_{j \in \mathcal{M} \setminus \{i\}} \frac{R_j}{\pi(i)}. \end{aligned} \quad (59)$$

The second inequality above follows from (58) and from the fact that $\mathbb{P}_i(\delta = j) \leq (R_j/\pi(i))$. The right-side of (59) goes to 0 since $R_i \rightarrow 0$, for each $i \in \mathcal{M}$. This proves Lemma 3. ■

The following result follows from a standard martingale convergence argument as in Lemma 5 in [18] and is omitted due to space constraints.

For any $0 < \rho < 1$, it holds that

$$\lim_{n \rightarrow \infty} \mathbb{P}_i \left\{ \max_{1 \leq m \leq n} \min_{j \in \mathcal{M} \setminus \{i\}} Z_{ij}(m) \geq n(d_i^* + 1 - \rho) \right\} = 0. \quad (60)$$

Combining the result in Lemma 3 and (60), we get for every $0 < \rho < 1$,

$$\mathbb{P}_i \left\{ N \leq \frac{-\rho \log R_i}{d_i^* + 1 - \rho} \right\} \rightarrow 0, \quad (61)$$

which is equivalent to the assertion of Lemma 2.

2) *Achievability Proof of Theorem 3 Without Condition (12):* Because the instantaneous control picked in (13) is a function only of the identity of the ML estimate of the hypothesis and not of the reliability of the estimate, e.g., the value of the posterior probability of the ML hypothesis, when the ML estimate is incorrect, the instantaneous control in (13) can be quite bad. This can happen with large probability especially when only a few observations are collected. Condition (12) essentially ensures

that when the ML hypothesis is incorrect, the control value of (13) will not be too bad. Consequently, (12) leads to a fast convergence of the ML estimate of the hypothesis to the true one when the ML estimation is used together with the control policy (13) at all times. Without (12), the convergence may not happen or even if it does, it may not be fast enough. This phenomenon is analogous to and is tightly connected to another one, which occurs in a somewhat more exacerbated form, in stochastic adaptive control [28] illustrating the failure of ML identification in closed-loop [29].

As previously mentioned at the end of Section IV-A, we slightly modify the control policy (13) by occasionally sampling from the uniform control independently of the identity of the ML hypothesis; this sparse sampling is used to guard against the event of incorrect ML estimation of the hypothesis. Precisely, for some $a > 1$, at times $k = \lceil a^l \rceil, l = 0, 1, \dots$, we let U_{k+1} be uniformly distributed on \mathcal{U} . At all other times, we still follow the control policy in (13). The stopping rule is still as in (14), and the final decision is still ML. Without loss of generality, we can assume that for every $i \neq j, i, j \in \mathcal{M}$, there exists a $u \in \mathcal{U}$ for which

$$D(p_i^u \| p_j^u) > 0, \quad (62)$$

otherwise, the probability of error can never be driven to zero. It now follows from (62) and the argument as in the proof of [18, Lemma 1] that for every $i \neq j$, and all n sufficiently large

$$\mathbb{P}_i \left\{ \sum_{k=1}^n L_k \right\} \leq e^{-b \frac{\log n}{\log a}},$$

where $L_k \triangleq \log(p_i^{U_k}(Y_k)/p_j^{U_k}(Y_k))$, for some $b > 0$, as we can only guarantee that $\mathbb{E}_i[e^{-(1/2)L_k} | \mathcal{F}_{k-1}] < 1$ for $\log n / \log a$ times in n time slots (precisely at those times when the control value is forced to be uniformly distributed). Let T be the earliest time such that the ML estimate of the hypothesis equals the true hypothesis for all time $k \geq T$. Then, we get that for all sufficiently large k ,

$$\mathbb{P}_i\{T > k\} \leq M \sum_{t \geq k} e^{-b \frac{\log t}{\log a}} \leq O(k^{-\gamma}) \quad (63)$$

for an arbitrary large γ when a is chosen to be sufficiently close to 1. Note that it was shown in [18, Lemma 1] that if (12) holds, then $\mathbb{P}_i\{T > k\}$ decays exponentially.

Our achievability proof of asymptotic optimality without (12), i.e., that the modified test satisfies (18) without imposing (12), follow closely the steps in the proof of [18, Lemma 2] under assumption (12). Due to space limitations, we shall just emphasize key steps and point out the difference from the proof when (12) is relaxed. To this end, we denote the maximizers in the denominator on the right-side of (18) by $q_i^*(u)$.

Referring to the stopping rule in (14), we see that the stopping time depends on the time needed for the Log-Likelihood Ratio (LLR) corresponding to the closest alternative hypothesis to cross the stopping threshold $-\log c$. Thus, the main idea is to show that the LLR per observation concentrates around the denominator on the right-side of (18) for the control policy described above. The key step in the proof of (18) deals with

the following decomposition for an arbitrary hypothesis $j \neq i$, where i is the true hypothesis,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n L_k &= \frac{1}{n} \sum_{k=1}^n \{L_k - \mathbb{E}_i[L_k | \mathcal{F}_{k-1}]\} \\ &\quad + \frac{1}{n} \sum_{k=1}^n \left\{ \mathbb{E}_i[L_k | \mathcal{F}_{k-1}] - \sum_u q_i^*(u) D(p_i^u \| p_j^u) \right\} \\ &\quad + \sum_u q_i^*(u) D(p_i^u \| p_j^u). \end{aligned} \quad (64)$$

The proof of the measure concentration then boils down to proving that the two averages of the bracketed $\{\}$ -quantities concentrate around 0 from the negative side with a sufficiently quick decay of the probability of non-concentration. In particular, it suffices to prove that the following two sequences of probabilities (as a function of n) go to zero sufficiently fast:

$$\mathbb{P}_i \left\{ \frac{1}{n} \sum_{k=1}^n \{L_k - \mathbb{E}_i[L_k | \mathcal{F}_{k-1}]\} < -\epsilon \right\}, \quad (65)$$

and

$$\mathbb{P}_i \left\{ \frac{1}{n} \sum_{k=1}^n \left\{ \mathbb{E}_i[L_k | \mathcal{F}_{k-1}] - \sum_u q_i^*(u) D(p_i^u \| p_j^u) \right\} < -\epsilon \right\}. \quad (66)$$

Note that the minimum value of the third term in the decomposition in (64) over $j \neq i$ is specifically the denominator on the right-side of (18).

The same argument leading to [18, Equation (5.10)] gives that (65) goes to zero exponentially. Also, (63) implies a polynomial decay of (66), as with probability 1,

$$\left| \sum_{k=1}^n \left\{ \mathbb{E}_i[L_k | \mathcal{F}_{k-1}] - \sum_u q_i^*(u) D(p_i^u \| p_j^u) \right\} \right| \leq C' \min(T, n) + C'' \log n,$$

for some constants C', C'' by virtue of fact that $q = q_i^*$ for each $k \geq T$, such that $k \neq \lceil a^l \rceil$, $l \geq 1$ (cf. the definition of T in above). This will lead us to [18, Equation (5.9)] but only with a polynomial decay (with an arbitrarily high degree γ in (63)) in the probability on the right-side of the equation. Nevertheless, the sufficiently quick polynomial decay in the probability therein still enables us to complete the steps at the beginning to proof of [18, Lemma 2] to eventually upper bound the asymptotes of the expected sample sizes to be (18).

3) *Proof of Theorem 4:* We first prove (22). Let i be the true hypothesis. For any $j \in \mathcal{M}$, $j \neq i$, consider the event

$$A_{n,j} = \{(y^n, u^n) : N_A = n, \delta = j\}. \quad (67)$$

Following the approach in [30], on the set $A_{n,j}$ we have the following set of inequalities,

$$\begin{aligned} \log \left(\frac{\pi(j) p_j(y^n, u^n)}{\pi(i) p_i(y^n, u^n)} \right) &\geq \log \left(\frac{\pi(j) p_j(y^n, u^n)}{\max_{i \neq j} \pi(i) p_i(y^n, u^n)} \right) \\ &\geq \log \left(\frac{(M-1)\pi(j)}{\bar{R}_j} \right). \end{aligned} \quad (68)$$

The last inequality above follows since the test ends at n and the stopping criteria must be met for the choice of the thresholds in (21). Thus,

$$\mathbb{P}_i \{A_{n,j}\} \leq \frac{\bar{R}_j}{(M-1)\pi(i)} \mathbb{P}_j \{A_{n,j}\}.$$

It now follows that

$$\begin{aligned} \mathbb{P}_i \{\delta = j\} &= \sum_{n=1}^{\infty} \mathbb{P}_i \{A_{n,j}\} \leq \frac{\bar{R}_j}{(M-1)\pi(i)} \sum_{n=1}^{\infty} \mathbb{P}_j \{A_{n,j}\} \\ &\leq \frac{\bar{R}_j}{(M-1)\pi(i)}. \end{aligned} \quad (69)$$

From the definition of R_j in (15), we then get that $R_j \leq \bar{R}_j$. The result holds for each $j \in \mathcal{M}$.

The last assertion of Theorem 4 pertaining to asymptotic optimality of the proposed test follows by considering yet another test with the stopping rule (21) being replaced by the following stopping rule with a single threshold

$$\log \left(\frac{p_{i_n}(y^n, u^n)}{\max_{j \neq i_n} p_j(y^n, u^n)} \right) \geq \log \left((M-1) \left(\max_{i \neq j} \frac{\pi(j)}{\bar{R}_i} \right) \right), \quad (70)$$

and with the same control and decision rule as those of the proposed test. It follows from (21) and (70) that the stopping time of this new test will always dominate (larger than) that of the proposed test a.s. Let us denote the two respective stopping times by N and N' . Since π has a full support, as $\max_{i \in \mathcal{M}} \bar{R}_i \rightarrow 0$, the single threshold on the right-side of (70) will go to infinity. By Theorem 3, this new test with the single threshold is asymptotically optimal, i.e., it satisfies, for every $i \in \mathcal{M}$,

$$\lim_{\max_i \bar{R}_i \rightarrow 0} -\frac{\mathbb{E}_i[N']}{\log c} \leq \frac{1}{\max_{q(u)} \min_{j \neq i} \sum_u q(u) D(p_i^u \| p_j^u)}, \quad (71)$$

where $c = (1/(M-1))(\min_{i \neq j} (\bar{R}_i/\pi(j)))$. On the other hand, it follows from (69) and the assumption in the statement of Theorem 4 that

$$\max_i \mathbb{P}_i \{\delta \neq i\} \leq \max_{i \neq j} \frac{\bar{R}_j}{\pi(i)} \leq K'c, \quad (72)$$

for a suitable constant K' . The aforementioned dominance, i.e., $N \leq N'$ a.s., and (72) along with (71) give that for every $i \in \mathcal{M}$,

$$\begin{aligned} \lim_{\max_i \bar{R}_i \rightarrow 0} -\frac{\mathbb{E}_i[N]}{\log \left(\max_k \mathbb{P}_k \{\delta \neq k\} \right)} &\leq \lim_{\max_i \bar{R}_i \rightarrow 0} -\frac{\mathbb{E}_i[N']}{\log c} \\ &\leq \frac{1}{\max_{q(u)} \min_{j \neq i} \sum_u q(u) D(p_i^u \| p_j^u)}. \end{aligned}$$

REFERENCES

- [1] D. A. Castañón, "Approximate dynamic programming for sensor management," in *Proc. 36th IEEE CDC*, San Diego, USA, Dec. 10–12, 1997, pp. 1202–1207.
- [2] G. K. Atia, V. V. Veeravalli, and J. A. Fuemmeler, "Sensor scheduling for energy-efficient target tracking in sensor networks," *IEEE Trans. Signal Process.*, vol. 59, pp. 4923–4937, Oct. 2011.

- [3] A. O. Hero, III, D. A. Castañón, D. Cochran, and K. Kastella, Eds., *Foundations and Applications of Sensor Management*, ser. Springer Series on Signals and Communication Technology. Berlin: Springer, 2008.
- [4] F. J. Anscombe, "Sequential medical trials," *J. Amer. Statist. Assoc.*, vol. 58, pp. 365–383, 1963.
- [5] Y. He and E. K. Chong, "Sensor scheduling for target tracking: A Monte Carlo sampling approach," *Digital Signal Process.*, vol. 16, no. 5, pp. 533–545, Sep. 2006, special Issue on DASP 2005.
- [6] V. Krishnamurthy and D. Djonin, "Structured threshold policies for dynamic sensor scheduling—a partially observed Markov decision process approach," *IEEE Trans. Signal Process.*, vol. 55, no. 10, pp. 4938–4957, Oct. 2007.
- [7] S. Nitinawarat, G. Atia, and V. V. Veeravalli, "Controlled sensing for hypothesis testing," in *Proc. IEEE ICASSP*, Kyoto, Japan, Mar. 25–30, 2012, pp. 5277–5280.
- [8] J. N. Tsitsiklis, "Decentralized detection by a large number of sensors," *Math. Contr. Signals Syst.*, vol. 1, pp. 167–182, 1988.
- [9] E. Berlekamp, "Block Coding With Noiseless Feedback," Ph.D. dissertation, MIT, Cambridge, 1964.
- [10] P. Mitran and A. Kavčić, "Error exponents for finite-hypothesis channel identification," *IEEE Trans. Inform. Theory*, vol. 52, pp. 245–255, Jan. 2006.
- [11] M. Hayashi, "Discrimination of two channels by adaptive methods and its application to quantum system," *IEEE Trans. Inform. Theory*, vol. 55, pp. 3807–3820, Aug. 2009.
- [12] A. Wald, *Sequential Analysis*. New York: Wiley, 1947.
- [13] A. Wald and J. Wolfowitz, "Optimum character of the sequential probability ratio test," *Ann. Math. Statist.*, vol. 19, pp. 326–339, 1948.
- [14] C. W. Baum and V. V. Veeravalli, "A sequential procedure for multiple hypothesis testing," *IEEE Trans. Inform. Theory*, vol. 40, pp. 1994–2007, Nov. 1994.
- [15] V. V. Veeravalli and C. W. Baum, "Asymptotic efficiency of a sequential multihypothesis testing," *IEEE Trans. Inform. Theory*, vol. 41, pp. 1994–1997, Nov. 1995.
- [16] V. P. Dragalin, A. G. Tartakovsky, and V. V. Veeravalli, "Multihypothesis sequential probability ratio tests—Part I: Asymptotic optimality," *IEEE Trans. Inform. Theory*, vol. 45, pp. 2448–2461, Nov. 1999.
- [17] V. P. Dragalin, A. G. Tartakovsky, and V. V. Veeravalli, "Multihypothesis sequential probability ratio tests—Part II: Accurate asymptotic expansions," *IEEE Trans. Inform. Theory*, vol. 46, pp. 1366–1383, Jul. 2000.
- [18] H. Chernoff, "Sequential design of experiments," *Ann. Math. Statist.*, vol. 30, pp. 755–770, 1959.
- [19] S. A. Bessler, Theory and Applications of the Sequential Design of Experiments, *k*-Actions and Infinitely Many Experiments: Part I—Theory Dept. of Statistics, Stanford Univ., Tech. Report no. 55, 1960.
- [20] M. V. Burnashev, "Sequential discrimination of hypotheses with control of observations," *Math. USSR-Izvestiya*, vol. 15, pp. 419–440, 1980.
- [21] M. Naghshvar and T. Javidi, "Active M-ary sequential hypothesis testing," in *Proc. IEEE Int. Symp. Infor. Theory*, Austin, TX, USA, June 13–18, 2010, pp. 1623–1627.
- [22] D. Blackwell, "Equivalent comparisons of experiments," *Ann. Math. Statist.*, vol. 24, pp. 265–272, 1953.
- [23] M. Naghshvar and T. Javidi, "Variable-length coding with noiseless feedback and finite messages," in *Proc. 44th Asilomar Conf. Signals, Syst. and Comput.*, Monterey, CA, USA, Nov. 7–10, 2010, pp. 317–321.
- [24] M. Naghshvar and T. Javidi, "Performance bounds for active sequential hypothesis testing," in *Proc. IEEE Int. Symp. Infor. Theory*, Saint Petersburg, Russia, July 31–August 5 2011, pp. 2666–2670.
- [25] M. Naghshvar and T. Javidi, "Active hypothesis testing: Sequentiality and adaptivity gains," in *Proc. the 46th Conf. Infor. Sciences and Syst.*, Princeton, NJ, USA, Mar. 21–23, 2012.
- [26] M. Loève, *Probability Theory II*. Berlin: Springer-Verlag, 1978.
- [27] D. P. Bertsekas, *Dynamic Programming and Optimal Control*. Belmont, MA: Athena Scientific, 2000.
- [28] P. R. Kumar and P. Varaiya, *Stochastic Systems: Estimation, Identification, and Adaptive Control*. NJ: Prentice-Hall, Inc., 1986.
- [29] V. Borkar and P. Varaiya, "Adaptive control of Markov chains, I: Finite parameter set," *IEEE Trans. Automat. Contr.*, vol. 24, pp. 953–958, 1979.
- [30] G. K. Atia and V. V. Veeravalli, "Controlled sensing for sequential multihypothesis testing," in *Proc. IEEE Int. Symp. Infor. Theory*, Cambridge, USA, Jul. 1–, 2012, pp. 2206–2210.



USA.

His research interests are in stochastic control, statistical signal processing, information and coding theory, communications, estimation and detection, and machine learning.

Dr. Nitinawarat was a co-organizer for the special session on Controlled Sensing for Inference at the 2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP); and chair for the session on Distributed Inference in Sensor Networks at the 49th Annual Allerton Conference on Communication, Control, and Computing (2011). He was a finalist for the best student-paper award at the IEEE International Symposium on Information Theory, which was held at Austin, TX, in 2010. During his Ph.D. study at the University of Maryland, he received graduate teaching fellowships in Fall 2007, Spring 2008, and Spring 2009.



George K. Atia (S'01–M'09) received the B.Sc. and M.Sc. degrees from Alexandria University, Egypt, in 2000 and 2003, respectively, and the Ph.D. degree from Boston University (BU), MA, USA, in 2009, all in electrical and computer engineering.

He joined the University of Central Florida in Fall 2012 where he is currently an Assistant Professor, Department of Electrical Engineering and Computer Science. From Fall 2009 to 2012, he was a Postdoctoral Research Associate at the Coordinated Science Laboratory at the University of Illinois at Urbana-Champaign. His research interests include statistical signal processing, stochastic control, wireless communications, detection and estimation theory, and information theory.

Dr. Atia is the recipient of many awards, including the Outstanding Graduate Teaching Fellow of the Year Award in 2003–2004 from the Department of Electrical and Computer Engineering, BU, the 2006 College of Engineering Dean's Award at the BU Science and Engineering Research Symposium, and the best paper award at the International Conference on Distributed Computing in Sensor Systems in 2008.



Venugopal V. Veeravalli (M'92–SM'98–F'06) received the B.Tech. degree (Silver Medal Honors) from the Indian Institute of Technology, Bombay, in 1985, the M.S. degree from Carnegie Mellon University, Pittsburgh, PA, USA, in 1987, and the Ph.D. degree from the University of Illinois at Urbana-Champaign, IL, USA, in 1992, all in electrical engineering.

He joined the University of Illinois at Urbana-Champaign in 2000, where he is currently a Professor in the Department of Electrical and Computer Engineering and the Coordinated Science Laboratory. He served as a Program Director for communications research at the U.S. National Science Foundation in Arlington, VA, from 2003 to 2005. He has previously held academic positions at Harvard University, Rice University, and Cornell University, and has been on sabbatical at MIT, IISc Bangalore, and Qualcomm, Inc. His research interests include distributed sensor systems and networks, wireless communications, detection and estimation theory, including quickest change detection, and information theory.

Prof. Veeravalli was a Distinguished Lecturer for the IEEE Signal Processing Society during 2010–2011. He has been an Associate Editor for Detection and Estimation for the IEEE TRANSACTIONS ON INFORMATION THEORY and for the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. Among the awards he has received for research and teaching are the IEEE Browder J. Thompson Best Paper Award, the National Science Foundation CAREER Award, and the Presidential Early Career Award for Scientists and Engineers (PECASE).