Quadratic detection of signals with drifting phase

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(Received 20 December 1989; accepted for publication 22 September 1990)

The problem of detecting a sinusoid with drifting phase in the presence of additive white Gaussian noise is considered. A Lorentzian signal model is used, in which the signal to be detected is modeled as a sinusoid whose phase is drifting with Brownian motion. A class of quadratic detectors that trade off coherent and noncoherent averaging of the received waveform is studied. The deflection ratio is used as a performance criterion, and the optimum quadratic detector structure as parametrized by the phase bandwidth is derived. Then, the performance relative to the optimum of a class of suboptical detectors called *m-order noncoherent detectors* is considered. It is shown that the best detector in this class performs nearly as well as the optimum quadratic detector. This is in sharp contrast with standard envelope detection whose performance is shown to degrade severely in the presence of phase drift. Simulated detection probabilities that verify this performance disparity are also presented.

PACS numbers: 43.60.Gk

INTRODUCTION

The reliable detection of weak acoustic signals typically requires the accumulation of signal energy over relatively long integration times. Although such signals may be nominally harmonic, as a practical matter the phase of such signals cannot be assumed to be perfectly stable throughout the integration time. Since the phase may drift substantially during the formation of the detection statistic, it is not realistic to model such signals as constant-phase sinusoids in the analysis and development of signal detection algorithms. On the other hand, if there is harmonic structure in the signal of interest, then it is inefficient not to exploit this structure (as would be the case, for example, if simple radiometry were used for detection). Thus it is of interest to consider detection algorithms that can trade off the coherent averaging that is optimum for constant-phase harmonic signals with the noncoherent averaging that is optimum for a purely stochastic signal, these two cases being the extremes of the behavior of a sinusoidal signal undergoing phase drift.

In this paper, we consider a class of quadratic detectors with which this tradeoff can be effected. We adopt a Lorentzian signal model, in which the signal to be detected is modeled as a sinusoid whose phase is drifting with Brownian motion. The severity of phase drift can then be parametrized by the variance parameter of this Brownian motion. It is assumed that this signal is observed in the presence of white Gaussian noise. Although the globally optimum detection statistic for such a problem is not quadratic (see, e.g., Poor[1]), we consider only quadratic detectors in this study due to the simplicity of their implementation relative to the infinite-dimensional estimator–correlator that is optimum. In view of the weak-signal, long-integration time scenario of interest, we adopt the deflection ratio[1] as a performance criterion by which to optimize detector structure, as is customary in the design of quadratic detectors (see, e.g., Baker[2]).

Within this setup, we first derive the optimum quadratic structure as parametrized by the phase bandwidth. We then consider the performance relative to this optimum of a class of suboptimal detectors, called *m-order noncoherent detectors*, that trade off coherent and noncoherent integration of the received waveform. This class of detectors is adapted from a similar application in the problem of on–off-keyed optical communications.[3] Here, we derive an optimum detector over this class (as a function of relative phase bandwidth) and show that this detector comes to within 2 dB of maximum deflection throughout the interesting range of parameters. This is in sharp contrast with standard envelope detection, whose performance is shown to degrade severely in the presence of phase drift. Simulated detection probabilities are also presented to verify this performance disparity.

This paper is organized as follows: In Sec. I, we present the signal and noise model of interest, and discuss the structure of the globally optimum (estimator–correlator) detection statistic, the conventional envelope detector, and the proposed family of m-order noncoherent detectors. The optimum quadratic detector is derived in Sec. II, and a deflection analysis of it and the proposed detector family is carried out in Sec. III. Section IV discusses the calculation of false-alarm probabilities and the corresponding threshold determination for the various detectors of interest, and detection probabilities estimated through computer simulation are presented. Finally, some concluding remarks and interesting open problems for further study are discussed in Sec. V.

I. BACKGROUND

A. The detection problem

The detection problem of interest can be modeled as a test between two statistical hypotheses, H_0 and H_1, described by

$$H_0\colon Y_t = N_t, \quad 0 \leqslant t \leqslant T$$

vs (1)

$$H_1\colon Y_t = A\cos(\omega_0 t + \Theta_t + \phi) + N_t, \quad 0 \leqslant t \leqslant T.$$

In this model, the signal $S_t = A\cos(\omega_0 t + \Theta_t + \phi)$, $0 \leqslant t \leqslant T$, has known frequency ω_0; nonzero amplitude A; and phase $\Theta_t + \phi$, $0 \leqslant t \leqslant T$, where ϕ is a random initial phase offset, uniformly distributed in the interval $[0,2\pi]$, and the phase drift process $\{\Theta_t;\ 0 \leqslant t \leqslant T\}$ is a Brownian motion, independent of ϕ. Note that the addition of the random phase ϕ renders the signal wide-sense stationary. The noise process $\{N_t;\ 0 \leqslant t \leqslant T\}$ is assumed to be a white Gaussian process, independent of the process $\{\Theta_t;\ 0 \leqslant t \leqslant T\}$ and the initial phase ϕ, and having two-sided power spectral density (PSD) $N_0/2$. Without significant loss of generality, it is assumed that the frequency of the signal satisfies $\omega_0 T/\pi = n$, for some large integer n.

Before proceeding further, it is convenient to normalize the hypothesis testing problem of Eq. (1) in the following way. We write Θ_t as $\Theta_t = \sqrt{2\beta}\, B_t$, $0 \leqslant t \leqslant T$, where β is a constant and $\{B_t;\ 0 \leqslant t \leqslant T\}$ is a unit Brownian motion (i.e., $\{B_t;\ 0 \leqslant t \leqslant T\}$; is a zero-mean Gaussian process with autocorrelation function $E\{B_t B_s\} = \min\{t,s\}$ for all $t,s \geqslant 0$). We then define the *signal-to-noise ratio* ρ, and the *relative phase bandwidth* (or the *time-bandwidth product*) ξ of the signal, as $\rho = A^2 T/N_0$ and $\xi = \pi\beta T$, respectively. Finally, we divide the observations by $\sqrt{N_0/2}$, and normalize the time so that the time period T is 1, to obtain the equivalent hypothesis pair (for convenience, the same symbols are retained after normalization):

$$H_0\colon Y_t = N_t, \quad 0 \leqslant t \leqslant 1$$

vs (2)

$$H_1\colon Y_t = \sqrt{2\rho}\cos(n\pi t + \Theta_t + \phi) + N_t, \quad 0 \leqslant t \leqslant 1.$$

With this normalization, the phase drift process is given by $\Theta_t = \sqrt{2\xi}\, B_t$, $0 \leqslant t \leqslant 1$, and the additive noise $\{N_t;\ 0 \leqslant t \leqslant 1\}$ has a two-sided PSD of 1. The normalized signal $S_t = \sqrt{2\rho} \times \cos(n\pi t + \Theta_t + \phi)$, $0 \leqslant t \leqslant 1$ has mean 0 and autocorrelation function given by (see, e.g., Ref. 4)

$$\begin{aligned} R_S(t,u) &= E\{S_t S_u\} \\ &= \rho\cos[n\pi(t-u)]\exp(-\xi\,|t-u|), \\ &\quad 0 \leqslant t,u \leqslant 1. \end{aligned} \tag{3}$$

The power spectrum of the signal, which can be obtained by taking the Fourier transform of Eq. (3), has a Lorentzian shape with center frequency $n\pi$, and with 3-dB bandwidth ξ as given below:

$$S_S(\omega) = 2\rho\xi/[(n\pi - \omega)^2 + \xi^2]. \tag{4}$$

To overcome the analytical difficulties encountered in dealing with white noise, we integrate the observations in Eq. (2) to get the following equivalent model (all stochastic integrals in the paper are defined in the mean-square sense):

$$H_0\colon X_t = W_t, \quad 0 \leqslant t \leqslant 1$$

vs (5)

$$H_1\colon X_t = \int_0^t S_u\, du + W_t, \quad 0 \leqslant t \leqslant 1,$$

where $X_t = \int_0^t Y_u du$, $0 \leqslant t \leqslant 1$, and where $\{W_t;\ 0 \leqslant t \leqslant 1\}$ is a unit Brownian motion independent of $\{B_t;\ 0 \leqslant t \leqslant 1\}$.

The model of Eq. (5) is the one that we shall analyze in the sequel. We turn first in this analysis to a brief description of optimum Bayesian detection in this model.

B. Neyman–Pearson optimum detection

The Neyman–Pearson optimum detector for the problem of Eq. (5), i.e., the scheme that has the minimum miss probability for a given level of false alarm probability, compares the likelihood ratio with a prespecified threshold to decide between the two hypotheses (see, Poor[1]). It is easy to see that the signal $\{S_t;\ 0 \leqslant t \leqslant 1\}$ satisfies the conditions

$$E\left\{\int_0^1 |S_t|\, dt\right\} \leqslant \infty,$$

and

$$\int_0^1 (S_t)^2\, dt < \infty \quad \text{with probability 1.}$$

Using these conditions along with the fact that the signal $\{S_t;\ 0 \leqslant t \leqslant 1\}$ and noise $\{N_t;\ 0 \leqslant t \leqslant 1\}$ are independent, it follows from Ref. 1 that the log-likelihood ratio for the problem of Eq. (5) has the following *estimator–correlator* structure:

$$\log \frac{dP_1}{dP_0}(X_0^1) = \int_0^1 \widehat{S}_t\, dX_t - \frac{1}{2}\int_0^1 (\widehat{S}_t)^2\, dt, \tag{6}$$

where X_0^t is a shorthand notation representing the set of observations: $\{X_u;\ 0 \leqslant u \leqslant t\}$; and $\widehat{S}_t \triangleq E_1\{S_t | X_0^t\}$ denotes the causal minimum-mean-square-error (MMSE) estimate of S_t given X_0^t, for $0 \leqslant t \leqslant 1$.

The Neyman–Pearson detection strategy would involve comparing the log-likelihood ratio of Eq. (6) with a threshold τ to decide between the two hypotheses. However, we should note that the estimator–correlator structure of Eq. (6) is only a *representation* of the log-likelihood ratio in terms of the estimate $\{\widehat{S}_t;\ 0 \leqslant t \leqslant 1\}$, which is in general quite difficult to determine when the signal is non-Gaussian (as is the case here).

An approximation to $\{\widehat{S}_t;\ 0 \leqslant t \leqslant 1\}$ can be obtained for the problem of interest by using the extended Kalman filter approach.[1] The approximation to the optimum detector thus obtained is a phase-lock loop (which tracks the time-varying phase) followed by a coherent detector. This detector structure has been proposed by Georghiades and Snyder[5] for the problem of sequence detection in the presence of phase noise in the context of optical communications. Unfortunately, this detector, although promising, is only an approximation to the optimum one; and the degree to which it serves as such is still an open question.

C. Noncoherent detection

As an alternative to the phase-tracking detectors described above, it is of interest to analyze the performance of detectors that do not need phase information. One such detector in common use is the standard noncoherent detector (SND), which is optimum for the problem of Eq. (5) when the phase is unknown but constant (i.e., when $\rho = 0$). The test statistic for this detector is given by

$$\Lambda_c = \left(\int_0^1 \cos(n\pi t) dX_t \right)^2 + \left(\int_0^1 \sin(n\pi t) dX_t \right)^2. \quad (7)$$

Note that this detector is potentially useful when the phase drift is minimal over the observation interval (i.e., when ξ is small) since it coherently averages the in-phase and quadrature components of the harmonic signal before extracting the energy. On the other hand, one would not expect this approach to work well when the phase drifts significantly over the observation interval, as can be the case in the long-integration time scenario of interest here. Thus it is of interest to consider a useful modification of this detector, referred to as the *m-order noncoherent detector* (mND), using the following test statistic:

$$\Lambda_{cm} = \sum_{i=1}^{m} \left[\left(\int_{(i-1)/m}^{i/m} \cos(n\pi t) dX_t \right)^2 + \left(\int_{(i-1)/m}^{i/m} \sin(n\pi t) dX_t \right)^2 \right], \quad (8)$$

where m is a positive integer characterizing the detector. Note that, for a fixed value of m, this detector trades off coherent integration over the intervals of length $1/m$, with noncoherent integration (i.e., post-detection integration) of the m coherently integrated segments. Thus, as we shall see below, the value of m can be optimized to yield the best trade-off of these two effects for a given degree of phase instability. Note that the SND is the $m = 1$ instance of the mND.

The m-order noncoherent approach has been proposed by Foschini *et al.*[3] for the problem of demodulating on–off-keyed optical signals in the presence of phase noise. This work demonstrated via simulation that this structure yields improved error-rate performance over the one of Eq. (7) in this problem. Here we will analyze this detector, together with others, in the model of Eq. (5), and we will derive an analytical method for choosing an optimum value of the integer parameter m as a function of the relative phase bandwidth ξ.

We note that the detection statistics described in Eqs. (7) and (8) are quadratic forms, which are of course natural methods of metering energy, regardless of degree of coherence. Quadratic detectors (or their bias-corrected counterparts) are well known to be locally optimum for the detection of stochastic signals in the presence of additive white Gaussian noise (see, e.g., Refs. 6, 7). These considerations motivate us to consider a class of detectors for the problem of Eq. (5) that have a general quadratic form described in the following section, and to optimize detection performance over this class. To do so, we will consider the *deflection criterion*, defined in the following section, as the criterion of optimality.

II. OPTIMUM QUADRATIC DETECTION FOR GENERAL SIGNALS

We consider quadratic detectors for the problem of Eq. (5) that decide between the hypotheses by comparing with a threshold τ the following quadratic form:

$$\Lambda = \int_0^1 \int_0^1 Q(t,u) dX_u \, dX_t, \quad (9)$$

where $Q(t,u)$ belongs to the Hilbert space $\mathscr{H}_2$ of square-integrable functions on $[0,1]^2$ on the real field. The inner product and norm on this Hilbert space are defined, respectively, by

$$\langle Q_1, Q_2 \rangle = \int_0^1 \int_0^1 Q_1(t,u) Q_2(t,u) dt\, du,$$

and

$$|Q_1| = \sqrt{\langle Q_1, Q_1 \rangle},$$

where $Q_1(t,u)$ and $Q_2(t,u)$ belong to $\mathscr{H}_2$.

Prior to finding the optimum quadratic detector, we need to choose a criterion for optimality. Ideally, one would like to find the quadratic detector that minimizes the miss probability for a given level of false alarm probability. However, due to the intractability of error probabilities for such non-Gaussian nonlinear problems, we use the deflection[2] as the optimality criterion. In particular, we select as optimum any $Q \in \mathscr{H}_2$ for which the deflection

$$H(\Lambda) = (E_1\Lambda - E_0\Lambda)^2 / \mathrm{var}_0(\Lambda), \quad (10)$$

is maximized. Note that the function H is a measure of signal-to-noise ratio, and its relation with the likelihood ratio is discussed in Ref. 8. The use of the deflection as an optimization criterion can be motivated by a number of arguments, not the least of which is that the optimum quadratic detector thus obtained is the locally optimum Neyman–Pearson detector for our problem.[9]

Before we proceed to find the optimum quadratic detector, we list some stochastic properties of Brownian motion that will be needed in our analysis. Proofs of these follow similarly to that of Proposition VI.D.1 in Ref. 1.

Proposition 1: A unit Brownian motion $\{W_t;\ 0 \leqslant t \leqslant 1\}$ has the following properties:

(1) $\int_0^1 \int_0^1 F(t,u) dW_t \, dW_u$ exists as a mean square (m.s.) integral if and only if F is square-integrable on $[0,1]^2$.

(2) If F is square-integrable on $[0,1]^2$, then

$$E\left\{ \int_0^1 \int_0^1 F(t,u) dW_t \, dW_u \right\} = \int_0^1 F(t,t) dt.$$

(3) If F and G are square-integrable on $[0,1]^2$, then

$$E\left\{ \int_0^1 \int_0^1 F(t,u) dW_t \, dW_u \int_0^1 \int_0^1 G(t,u) dW_t \, dW_u \right\}$$
$$= \int_0^1 \int_0^1 F(t,t) G(u,u) dt\, du + \int_0^1 \int_0^1 F(t,u) G(t,u) dt\, du + \int_0^1 \int_0^1 F(t,u) G(u,t) dt\, du.$$

We may now state the following result.

Theorem 1: For the detection problem of Eq. (5), the deflection criterion is maximized over $\mathscr{H}_2$ by $Q_{\rm opt}(t,u) = R_S(t,u), 0\leqslant t,u\leqslant 1$, and the corresponding maximum value of the deflection is $H_{\rm opt} = \frac{1}{2}\langle R_S, R_S\rangle$.

Proof: We begin by finding an expression for the deflection of a general quadratic detector as defined in Eq. (9) when it is used to discriminate between the hypotheses of Eq. (5).

By property (2) of Proposition 1, we have that

$$E_0\{\Lambda\} = E\left\{\int_0^1 \int_0^1 Q(t,u)dW_t\, dW_u\right\} = \int_0^1 Q(t,t)dt. \tag{11}$$

The independence of $\{S_t; 0\leqslant t\leqslant 1\}$ and $\{W_t; 0\leqslant t\leqslant 1\}$ gives us

$$\begin{aligned} E_1\{\Lambda\} - E_0\{\Lambda\} &= E\left\{\int_0^1 \int_0^1 Q(t,u)S_tS_u\, dt\, du\right\} \\ &= \int_0^1 \int_0^1 Q(t,u)R_S(t,u)dt\, du \\ &= \langle Q,R_S\rangle. \end{aligned} \tag{12}$$

From property (3) of Proposition 1, we then have

$$\begin{aligned} E_0\{\Lambda^2\} &= E\left\{\left[\int_0^1 \int_0^1 Q(t,u)dW_t\, dW_u\right]^2\right\} \\ &= \left[\int_0^1 Q(t,t)dt\right]^2 + \langle Q,Q\rangle + \langle Q,Q^*\rangle, \end{aligned} \tag{13}$$

where $Q^*(t,u) \triangleq Q(u,t),\ 0\leqslant t,u\leqslant 1$. From Eqs. (11) and (13), we get a simple expression for the variance of Λ under H_0 as

$$\mathrm{var}_0(\Lambda) = \langle Q,Q\rangle + \langle Q,Q^*\rangle. \tag{14}$$

Substituting Eqs. (14) and (12) into Eq. (10), we have

$$H = [\langle Q,R_S\rangle]^2/[\langle Q,Q\rangle + \langle Q,Q^*\rangle]. \tag{15}$$

By invoking the symmetry of R_S, H can be written as

$$H = \frac{1}{2}\frac{[\langle Q+Q^*, R_S\rangle]^2}{\langle Q+Q^*, Q+Q^*\rangle}. \tag{16}$$

Applying the Schwartz inequality thus yields

$$H\leqslant \tfrac{1}{2}\langle R_S,R_S\rangle,$$

with equality if and only if

$$Q(t,u) + Q^*(t,u) = \alpha R_S(t,u),$$

for some nonzero real scalar α. In particular, a solution that maximizes the deflection is given by the symmetric function $Q_{\rm opt}(t,u) = R_S(t,u)$, and the corresponding deflection is $H_{\rm opt} = \frac{1}{2}\langle R_S,R_S\rangle$.

Note that Theorem 1 remains valid for arbitrary signals in the model of Eq. (5). All that is required is the interchange of the integration and expectation in Eq. (12), a condition for which mean-square continuity is sufficient. Thus this result is applicable to a much more general class of problems than that considered in this paper.

III. DEFLECTION ANALYSIS

In this section, we compare the deflection of the optimum quadratic detector with that of the noncoherent detectors introduced in Sec. I C, under the model described in Sec. I A. The purpose of this exercise is to investigate how much can be gained, in terms of deflection, by using the optimum quadratic detector instead of the simpler noncoherent detectors. It should be noted that the relative performance of the detectors measured using the deflection criterion could be different from that measured using error probabilities. However, as noted previously, the error-probability analysis of general quadratic detectors is prohibitively difficult when the signal is non-Gaussian (as is the case here).

Henceforth, in this paper we shall make the "narrowband" assumption that $n\gg\xi$, i.e., that the center frequency of the signal is very much larger than the bandwidth of the signal spectrum. With this assumption, we may obtain approximate expressions for the various quantities of interest that are independent of n. It should be noted that the very same expressions could also be obtained by using a baseband envelope detection approximation at the outset (as is done, for example, in Ref. 3).

We first note that the deflection of the optimum detector can be evaluated as follows:

$$\begin{aligned} H_{\rm opt} &= \frac{1}{2}\int_0^1 \int_0^1 [R_S(t,u)]^2\, dt\, du \\ &= \frac{\rho^2}{2}\int_0^1 \int_0^1 \cos^2[n\pi(t-u)] \\ &\qquad \times\exp(-2\xi\,|t-u|)dt\, du \\ &= \frac{\rho^2}{2}\int_{-1}^1 (1-|\tau|)\cos^2(n\pi\tau)\exp(-2\xi\,|\tau|)d\tau \\ &= \rho^2\int_0^1 (1-\tau)\cos^2(n\pi\tau)\exp(-2\xi\tau)d\tau \\ &= \frac{\rho^2}{2}\left(\int_0^1 (1-\tau)\exp(-2\xi\tau)d\tau + \int_0^1 (1-\tau)\right. \\ &\qquad \left.\times\cos(2n\pi\tau)\exp(-2\xi\tau)d\tau\right). \end{aligned} \tag{17}$$

It can be shown straightforwardly that the second integral on the right-hand side of Eq. (17) is negligible for $n\gg\xi$, and so we get the following approximation in this case:

$$\begin{aligned} H_{\rm opt} &\approx \frac{\rho^2}{2}\int_0^1 (1-\tau)\exp(-2\xi\tau)d\tau \\ &= \rho^2[2\xi + \exp(-2\xi) - 1]/8\xi^2. \end{aligned} \tag{18}$$

We now turn to the evaluation of the deflection of the m-order noncoherent detector of Eq. (8). Under H_0, we have

$$\begin{aligned} \Lambda_{cm} &= \sum_{i=1}^m \left[\left(\int_{(i-1)/m}^{i/m} \cos(n\pi t)dW_t\right)^2 + \left(\int_{(i-1)/m}^{i/m} \sin(n\pi t)dW_t\right)^2\right] \\ &= \sum_{i=1}^m (N_{1i})^2 + (N_{2i})^2, \end{aligned}$$

where

$$N_{1i} = \int_{(i-1)/m}^{i/m} \cos(n\pi t)dW_t$$

and

$$N_{2i} = \int_{(i-1)/m}^{i/m} \sin(n\pi t)\,dW_t.$$

On assuming that $n \gg m$, the random variables N_{1i}, $i = 1,...,m$, and N_{2i}, $i = 1,...,m$, can be shown to be well approximated by independent normal random variables with means 0 and variances $1/2m$. This approximation is exact if n/m is integral. Thus we have

$$\begin{aligned}\mathrm{var}_0(\Lambda_{em}) &\approx \sum_{i=1}^{m} \mathrm{var}[(N_{1i})^2] + \mathrm{var}[(N_{2i})^2] \\ &= \sum_{i=1}^{m} \frac{2}{(2m)^2} + \frac{2}{(2m)^2} \\ &= 1/m. \end{aligned} \tag{19}$$

The independence of $\{S_t;\, 0 \leqslant t \leqslant 1\}$ and $\{W_t;\, 0 \leqslant t \leqslant 1\}$ yields

$$\begin{aligned} &E_1\{\Lambda_{em}\} - E_0(\Lambda_{em}\} \\ &= \sum_{i=1}^{m} E\left[\left(\int_{(i-1)/m}^{i/m} \cos(n\pi t)S_t\,dt\right)^2 + \left(\int_{(i-1)/m}^{i/m} \sin(n\pi t)S_t\,dt\right)^2\right] \\ &= \sum_{i=1}^{m} \int_{(i-1)/m}^{i/m} \int_{(i-1)/m}^{i/m} \cos[n\pi(t-u)]R_S(t,u)\,dt\,du \\ &= \rho \sum_{i=1}^{m} \int_0^{1/m} \int_0^{1/m} \cos^2[n\pi(t-u)] \times \exp(-\xi|t-u|)\,dt\,du \\ &= 2\rho m \int_0^{1/m} \left(\frac{1}{m} - \tau\right) \cos^2(n\pi\tau)\exp(-\xi\tau)\,d\tau \\ &= \frac{2\rho}{m} \int_0^1 (1-\tau)\cos^2\left(\frac{n\pi\tau}{m}\right) \exp\left(-\frac{\xi\tau}{m}\right) d\tau \\ &\approx \frac{\rho}{m} \int_0^1 (1-\tau)\exp\left(-\frac{\xi\tau}{m}\right) d\tau \quad \left(n \gg \frac{\xi}{m}\right) \\ &= \frac{\rho}{m}\left(\frac{\xi/m + \exp(-\xi/m) - 1}{(\xi/m)^2}\right). \end{aligned} \tag{20}$$

From Eqs. (19) and (20), we obtain an approximate expression for the deflection of the detector based on Λ_{em} as

$$H(\Lambda_{em}) \approx \frac{\rho^2}{m} \frac{[\xi/m + \exp(-\xi/m) - 1]^2}{(\xi/m)^4}. \tag{21}$$

With $\xi_m \triangleq \xi/m$, we then have

$$H(\Lambda_{em}) \approx (\rho^2/\xi)\{[\xi_m + \exp(-\xi_m) - 1]^2/\xi_m^3\}.$$

Maximizing this expression with respect to ξ_m, yields that the optimum value of ξ_m is $\xi_m^* = 2.141926$. Hence, an appropriate choice of m for fixed ξ is given by

$$m^* = \begin{cases} 1, & \text{if } \xi \leqslant \xi_m^*, \\ \text{integer nearest to } \xi/\xi_m^*, & \text{otherwise.} \end{cases}$$

Thus we see that the deflection criterion gives us a very simple way of determining an optimum m-order noncoherent detector. This is in contrast with error-probability based criteria, for which an optimum value of m can be determined only by simulation. (It should be noted, of course, that m^* could be different from the value of m that yields the noncoherent detector with the smallest miss probability for a given value of false-alarm probability and a given value of ξ.)

Substituting the value $m = 1$ into Eq. (21) yields the deflection of the standard noncoherent detector based on Λ_e, viz.,

$$H(\Lambda_e) = \rho^2\{[\xi + \exp(-\xi) - 1]^2/\xi^4\}. \tag{22}$$

Table I lists the deflections of the SND and the m^*ND relative to that of the optimum quadratic detector, expressed in dB (i.e., $10\log_{10}$ of the ratio). It is apparent from the table that the m^*ND performs substantially better than the SND at high values of relative phase bandwidth ξ. This suggests that the error probability performance of the m^*ND should be much better than that of the SND for significantly drifting phase. (In the next section, we consider this issue.) We also note that the deflection of the m^*ND is within 2 dB of the optimum deflection over a very wide range of ξ values, suggesting that most of the performance that can be obtained with quadratic detection is achieved by the m^*ND.

Figure 1 shows curves of the deflection of the mND relative to that of the SND versus m for a few values of ξ. The dependence of m^* on ξ is clear from these curves. Figure 2 plots the same relative deflection versus ξ for various values of m. We see that all these curves flatten out for large values of ξ. This shows that a given mND performs uniformly better than the SND for all values of ξ greater than some threshold value, indicating the robustness of the mND for large values of ξ.

IV. ERROR PROBABILITY EVALUATION

In this section, we compare the error probabilities of the three quadratic detectors discussed in the paper. All of these

TABLE I. Deflections of the SND and the m^*ND relative to that of the optimum quadratic detector as a function of ξ.

		Relative deflection (dB)	
ξ	m^*	SND	m^*ND
0.10	1	− 0.002	− 0.002
1.10	1	− 0.24	− 0.24
1.32	1	− 0.34	− 0.34
1.58	1	− 0.47	− 0.47
1.91	1	− 0.63	− 0.63
2.29	1	− 0.85	− 0.85
2.75	1	− 1.12	− 1.12
3.31	2	− 1.46	− 1.28
3.98	2	− 1.86	− 1.33
4.79	2	− 2.32	− 1.44
5.75	3	− 2.84	− 1.52
6.92	3	− 3.41	− 1.58
8.32	4	− 4.02	− 1.63
10.00	5	− 4.67	− 1.69
12.02	6	− 5.35	− 1.72
14.45	7	− 6.05	− 1.75
17.38	8	− 6.77	− 1.77
20.89	10	− 7.50	− 1.80
25.12	12	− 8.24	− 1.81
30.20	14	− 9.00	− 1.83
36.31	17	− 9.76	− 1.84
43.65	20	− 10.53	− 1.85
52.48	24	− 11.30	− 1.86
63.10	29	− 12.08	− 1.87
75.86	35	− 12.87	− 1.87
91.20	42	− 13.65	− 1.88

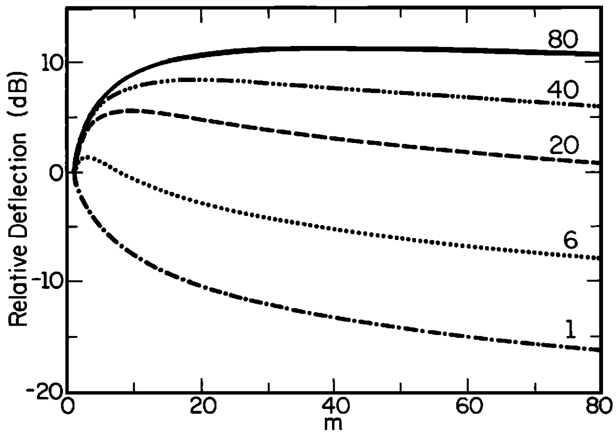

FIG. 1. Deflection of the mND relative to that of the SND as a function of m. Each curve corresponds to a distinct value of ξ (indicated next to the curve).

detectors compare a quadratic test statistic with a threshold τ to distinguish between the two hypotheses. Hypothesis H_1 is chosen if the statistic exceeds τ, and H_0 is chosen otherwise. We use the Neyman–Pearson approach, i.e., we specify the false alarm probabilty (P_F), and calculate the thresholds that yield this value of P_F for the three detectors. As we shall see, the threshold calculation can be done analytically since the observation is Gaussian under H_0. These thresholds can then be used in the estimation of the miss probabilities by simulation. In this latter analysis, we will omit the optimum quadratic detector since the deflection analysis suggests that it performs similarly to the optimum mND, and since its accurate simulation is complicated significantly by its infinite-dimensional structure.

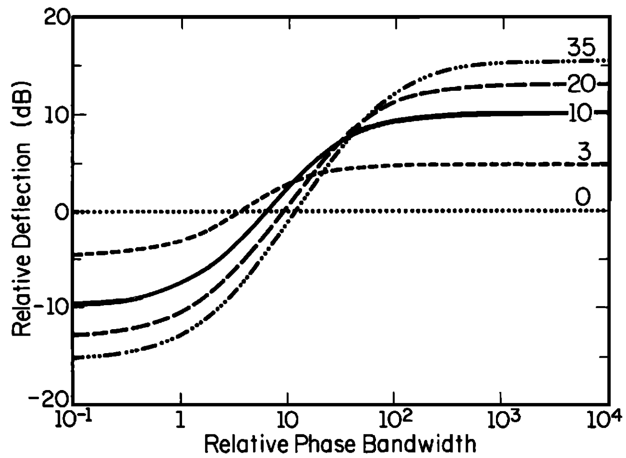

FIG. 2. Deflection of the mND relative to that of the SND as a function of ξ. Each curve corresponds to a distinct value of m (indicated next to the curve).

A. Threshold calculation

We first consider the false-alarm probabilities of the three detectors of interest.

1. Optimum quadratic detector

The false-alarm probability for the optimum quadratic detector is given by

$$P_F = P_0(\Lambda_{\rm opt} > \tau) = P_0(\Lambda_{\rm opt}/\rho > \tau/\rho). \tag{23}$$

Denote by $\phi_0(\omega) \triangleq E\{\exp(i\omega\Lambda_{\rm opt}/\rho)|H_0\}$ the characteristic function of $\Lambda_{\rm opt}/\rho$ under H_0, and define $\tilde{\tau} \triangleq \tau/\rho$. Then, from Ref. 10, we can write that

$$P_F = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{|\phi_0(\omega)|}{\omega} \sin[\angle\phi_0(\omega) - \omega\tilde{\tau}]d\omega, \tag{24}$$

where $\angle\phi_0(\omega)$ denotes the argument of $\phi_0(\omega)$. Under H_0,

$$\begin{aligned}\frac{\Lambda_{\rm opt}}{\rho} &= \frac{1}{\rho}\int_0^1\int_0^1 R_S(t,u)dW_t\,dW_u \\ &= \int_0^1\int_0^1 \cos[n\pi(t-u)] \\ &\quad\times\exp(-\xi|t-u|)dW_t\,dW_u.\end{aligned} \tag{25}$$

The symmetric kernel $\exp(-\xi|t-u|)$ has the following eigenfunction expansion on $[0,1]^2$:

$$\exp(-\xi|t-u|) = \sum_{k=1}^\infty \lambda_k\psi_k(t)\psi_k(u). \tag{26}$$

The eigenvalues and eigenfunctions are given in Ref. 11. In particular, we define the sequence $\{\beta_k;\, k=1,2,...\}$ by

$$(k-1)\pi < \beta_k < k\pi, \quad k=1,2,...,$$

and

$$\cot(\beta_k/2) = \beta_k/\xi, \quad \text{if } k \text{ is odd},$$
$$\tan(\beta_k/2) = -\beta_k/\xi, \quad \text{if } k \text{ is even}.$$

Then, the eigenvalues of Eq. (26) are given by

$$\lambda_k = 2\xi/(\xi^2+\beta_k^2), \quad k=1,2,...,$$

and the corresponding normalized eigenfunctions are

$$\psi_k(t) = \frac{\phi_k(t)}{[\int_0^1|\phi_k(t)|^2\,dt\,]^{1/2}}, \quad 0\leqslant t\leqslant 1, \quad k=1,2,...,$$

where

$$\phi_k(t) = \xi\beta_k^{-1}\sin(\beta_k t) + \cos(\beta_k t),$$
$$0\leqslant t\leqslant 1, \quad k=1,2,... .$$

Substituting Eq. (26) in Eq. (25), we get

$$\begin{aligned}\frac{\Lambda_{\rm opt}}{\rho} &= \int_0^1\int_0^1 \cos[n\pi(t-u)] \\ &\quad\times\sum_{k=1}^\infty \lambda_k\psi_k(t)\psi_k(u)dW_t\,dW_u\end{aligned}$$

$$= \sum_{k=1}^{\infty} \lambda_k \left[\left(\int_0^1 \psi_k(t)\cos(n\pi t)\,dW_t \right)^2 + \left(\int_0^1 \psi_k(t)\sin(n\pi t)\,dW_t \right)^2 \right].$$

[It can be shown that the condition required to interchange the integral and the summation above is that $E\{(\int_0^1\int_0^1 |dW_t||dW_u|)^2\} < \infty$, and the validity of this condition can easily be verified for Brownian motion.]

The eigenvalues λ_k decrease to zero with order k^{-2}, and hence we can approximate the infinite sum above by a finite sum of the first N terms, for sufficiently large N. Also, define

$$n_{1k} = \int_0^1 \psi_k(t)\cos(n\pi t)\,dW_t, \quad k = 1,2,...,N$$

and

$$n_{2k} = \int_0^1 \psi_k(t)\sin(n\pi t)\,dW_t, \quad k = 1,2,...,N.$$

Then, as argued before, for large n, n_{1k}, $k = 1,2,...,N$, and n_{2k}, $k = 1,2,...,N$, are well approximated by a set of independent normal random variables with means 0 and variances $\frac{1}{2}$. We can thus approximate

$$\frac{\Lambda_{\text{opt}}}{\rho} \approx \sum_{k=1}^{N} \lambda_k (n_{1k}^2 + n_{2k}^2),$$

$$\phi_0(\omega) = E\{\exp(j\omega\Lambda_{\text{opt}}/\rho)\} \approx \prod_{k=1}^{N} E\{\exp(j\omega\lambda_k n_{1k}^2)\}E\{\exp(j\omega\lambda_k n_{2k}^2)\} = \prod_{k=1}^{N} (1 - j\omega\lambda_k)^{-1},$$

$$|\phi_0(\omega)| \approx \prod_{k=1}^{N} (1 + \omega^2\lambda_k^2)^{-2}, \tag{27}$$

and

$$\angle\phi_0(\omega) \approx \sum_{k=1}^{N} \tan^{-1}(\omega\lambda_k). \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (24), we obtain an expression that can be used to calculate the value of $\tilde{\tau}$ corresponding to the prespecified false alarm probability.

2. Noncoherent detectors

For the mND detector, the quadratic test statistic is given under H_0 by

$$\Lambda_{em} = \sum_{i=1}^{m} (N_{1i})^2 + (N_{2i})^2,$$

where, for $n \gg m$, the random variables N_{1i}, $i = 1,...,m$, and N_{2i}, $i = 1,...,m$ are well approximated by independent random variables with means 0 and variances $1/2m$. Hence, $2m\Lambda_{em}$ has a chi-squared distribution with $2m$ degrees of freedom. If we denote by $\chi_{2m}(\cdot)$ the distribution function of chi-squared random variable with $2m$ degrees of freedom, then

$$P_F = P_0(\Lambda_{em} > \tau) = 1 - \chi_{2m}(2m\tau). \tag{29}$$

Thus, we have a relationship between P_F and τ, as desired.

B. Miss probabilities by simulation

The probability of miss for the detector with quadratic test statistic Λ and threshold τ is given by

$$P_M = P_1(\Lambda < \tau).$$

The closed-form computation of this quantity is difficult in the case of a Gaussian signal (see, e.g., Ref. 12), and prohibitive in the case of a non-Gaussian signal as we have here. Thus we estimate the miss probabilities of the two detectors of interest (the standard noncoherent detector and the m-order noncoherent detector) at various values of ξ and ρ via simulation.

We simulate the observations under H_1 by generating sample paths of the phase drift and additive noise processes. For each sample path of the observation process $\{Y_t; 0 \leqslant t \leqslant 1\}$, we compute the quadratic test statistics, compare these with the threshold values as calculated in the previous section, and use a relative frequency count to estimate the miss probabilities. This process can be facilitated as follows. Under H_1,

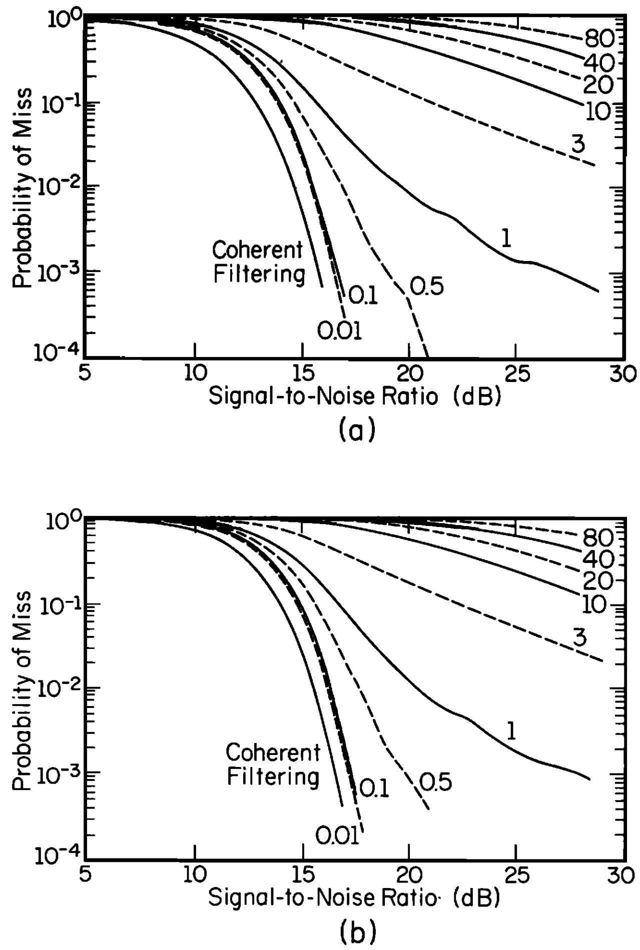

FIG. 3. Miss probabilities of the SND as function of ρ for several values of ξ (indicated next to each curve) and for two values of false alarm probability: (a) $P_F = 10^{-3}$ and (b) $P_F = 10^{-4}$.

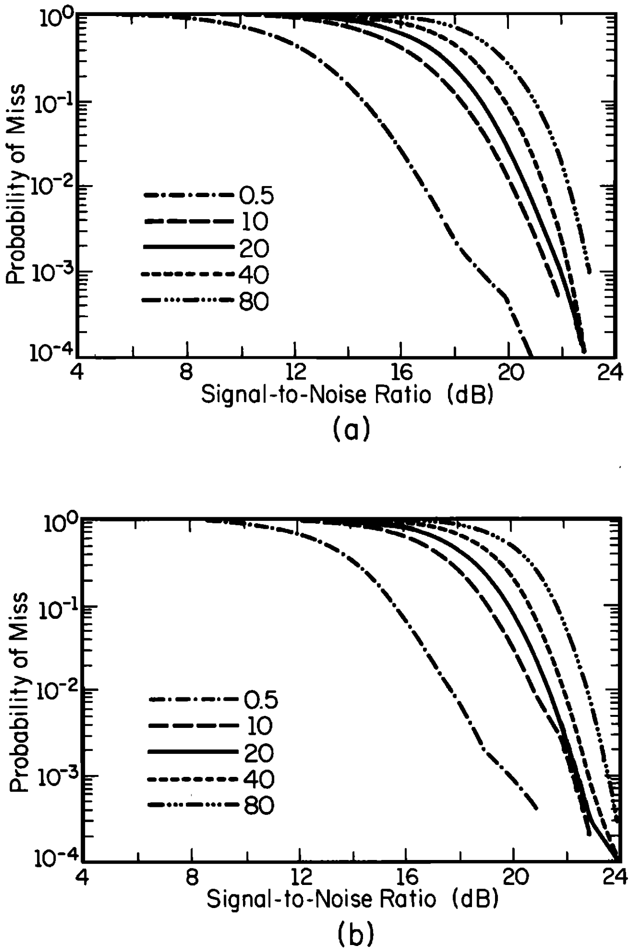

FIG. 4. Miss probabilties of the m^*ND as function of ρ for several values of ξ (indicated next to each curve) and for two values of false alarm probability: (a) $P_f = 10^{-3}$ and (b) $P_f = 10^{-4}$.

$$\Lambda_{em} = \sum_{i=1}^{m} \left[\left(\int_{(i-1)/m}^{i/m} \cos(n\pi t) dX_t \right)^2 + \left(\int_{(i-1)/m}^{i/m} \sin(n\pi t) dX_t \right)^2 \right],$$

where $dX_t = S_t\, dt + dW_t$. Substituting directly and expanding, we get

$$\Lambda_{em} = \sum_{i=1}^{m} (s_{1i} + N_{1i})^2 + (s_{2i} + N_{2i})^2,$$

where the random variables s_{1i}, s_{2i}, N_{1i}, and N_{2i} are approximated assuming that n is large. In particular,

$$s_{1i} \approx \int_{(i-1)/m}^{i/m} \sqrt{\frac{\rho}{2}} \cos[\Theta_t + \phi]\, dt,$$

$$s_{2i} \approx -\int_{(i-1)/m}^{i/m} \sqrt{\frac{\rho}{2}} \sin[\Theta_t + \phi]\, dt,$$

and the random variables N_{1i}, $i = 1,...,m$, and N_{2i}, $i = 1,...,m$, are approximated by independent normal random variables with means 0 and variances $1/2m$.

Figures 3 and 4 describe the simulation results. Two values of false alarm probability (10^{-3} and 10^{-4}) were chosen; and the corresponding miss probabilities for the two detectors were estimated for various values of ρ and ξ. The miss probabilities were estimated using only 10^4 independent trials and hence cannot be expected to be better than 10% accurate for values smaller than 10^{-2}. Nevertheless, the figures do show the trends predicted by the deflection of analysis of Sec. III.

In Fig. 3, we see that for very small values of ξ (i.e., negligible phase drift), the SND performs approximately 0.75 dB worse than the coherent detector with known phase, which is consistent with the well-known result for noncoherent detection with constant but unknown phase. For larger phase drifts (i.e., $\xi > 3$), we observe that the performance of the SND decreases drastically as ξ is increased. This is in contrast with the performance of the m^*ND detector which performs reasonably well even at large values of ξ. In fact, we see that for the range of values of ξ considered, the performance gain in error probability is as large as 10 dB [consider, for example, the curves for $\xi = 80$ in Fig. 3(a) and (b)]. The deflection analysis is in fact pessimistic in its prediction of the performance gain.

V. CONCLUSIONS

In this paper, we have considered the problem of detecting a sinusoid with drifting phase, in the presence of additive white Gaussian noise. We have shown that, in the class of quadratic detectors for this problem, the m^*-order noncoherent detector performs nearly as well as the optimum quadratic detector when the performance measure is the deflection ratio. Also, using the deflection criterion, we have derived a very simple way to determine the best m-order noncoherent detector. Finally, the analytical and simulation results in the paper show that the m^*-order noncoherent detector promises substantial performance gain over the standard noncoherent detector for large values of relative phase bandwidth.

It should be noted that the deflection analysis considered here uses only the second-order statistics of the signal corrupted by phase noise. Thus the analysis could be easily repeated for other phase drift models provided these models render the signal wide sense stationary and facilitate the computation of the signal autocorrelation function. Also, the detection problem considered here is relevant in applications of detecting acoustic signals under water, where the limiting background noise may very well not be accurately modeled as white Gaussian noise (see, e.g., Refs. 13–15). Thus, the extension of the analysis in this paper to other noise scenarios is an interesting topic for further study as well.

ACKNOWLEDGMENTS

This research was supported by the U.S. Office of Naval Research under Grant N00014-89-J-1321.

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